Productivity Fluctuation, Bank Lending and Nominal Interventions on Product Markets

Tianxi Wang*

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Abstract

Typically, central banks intervene with asset markets. This paper shows that if the productivity fluctuation is sufficiently large, the following nominal intervention on a product market is non-neutral: Whenever the negative productivity shock hits, the central bank produces fiat money to buy the product; and later retires money via product-money exchanges. The intervention increases the nominal price fluctuation and the bad-state profit margin of bank lending. The two have opposite effects for efficiency. The net effect depends on how the intervention is wound down. Furthermore, banks' money creation reduces lending rates by leveraging up the return to holding fiat money.

Key words: Intervention with product markets, price fluctuation, nominal profit margin, nominal non-neutrality, money creation

*University of Essex. wangt@essex.ac.uk.

1 Introduction

The question of how a central bank can affect real economic performance by simply printing a certain type of paper (or a clicking a mouse in modern times) has long fascinated the economics profession. While traditionaly the literature focuses on fiat money's role as a medium of exchange, a recent strand of literature finds it fruitful to focus on its role in the creating and lending of bank money and to have bank money serve the role of media of exchange.[1] This approach, indeed, captures the following basic facts in a modern advanced economy: The major means of payment for goods or services is not fiat money, but bank money; and the majority of fiat money is held by banks to facilitate the lending of bank money.[2] Taking this approach, the present paper considers a new possibility of central banking. Typically central banks intervenes with asset markets and this type of interventions are what the literature has considered. However, this paper shows that if the productivity fluctuation is large enough, a nominal intervention with a product market produces real effects, and with a proper design, improves real economic efficiency.

The mechanism builds on the new effects that this paper finds of the productivity fluctuation on the nominal price level and real-sector borrowers' default on nominal bank loans. If *one focuses on fiat money's role as a means of payment*, he naturally comes to the accounting equation that the nominal output – the product of the nominal price level and the real output – is equal to the quantity of fiat money times the velocity of circulation, the very equation that underpins the Quantity Theory of Money (QTM). It has two implications. First, if a negative productivity shock hits so the real output falls, the nominal price level rises, unless money circulates slower, for which there is no reason, or the government intervenes to reduce the quantity of fiat money. Therefore, the nominal price levels fluctuates *whenever* the productivity does, ex government

[1] See Bianchi and Bigio (2017) and Wang (2019, 2022); a detailed review of the literature is given later.

[2] For example, between June 2016 and June 2017 (i.e. before the crisis), according to Rule (2015), on the liability side of Bank of England, there are about as much of reserves and cash ratio deposits as the Bank's notes. The former is held by commercial banks only, which also hold a substantial fraction of notes. Therefore, more than half of the fiat money that the Bank creates is held by commercial banks.

interventions. Second, because the nominal price level rises in proportion upon the negative productivity shock, the nominal revenue of real-sector producers is little changed. So is their ability to repay nominal bank loans. That is, real-sector borrowers *never* default on nominal loans under *any* negative productivity shock. In this paper, these two implications hold no longer, and that opens a door for a product-market nominal operation to be non-neutral. The reason for their invalidation is that in this paper, it is bank money that serves as the means of payment for the product, and, because the quantity of bank money is adjusted by banks based on the economic condition, the accounting equation no longer holds.

The model economy is in an infinite horizon, populated by a continuum of banks, entrepreneurs, and workers. At the beginning of each period, banks ceate bank money and lend it to entrepreneurs; to fix the idea, let bank money be in the form of banknotes (see Section 2 for more details). Entrepreneurs use bank money as a means of wage payment to employ workers for the production of the consumption good, corn. The producivity can be high or low. The uncertainty resolves at the end of the period, when entrepreneurs produce corn. They then sell corn for bank money and use it to repay the loans. Among the suppliers of bank money – the buyers of corn – are the workers who have received their wages in bank money. In addition, banks pay dividends in bank money, which their shareholders use to buy corn. While fiat money is not needed as a means of payment for worker's labor or corn, it is needed for the lending of bank money. Banks are subject to a scale constraint: The quantity of bank money that a bank lends out cannot exceed a multipleχ of its fiat money position. If this multiple is 1, it is 100%-reserve banking and banks do not create money; the multiple therefore represents the scale of banks' money creation. Holding fiat money is costly due to time preference. The gain is that one unit of fiat money enables the bank to lend out χ units of bank money and thus earns χ times of the lending profit margin.

We find that contary to the QTM, the nominal price fluctuates if and only if the scale of the productivity fluctuation is above a threshold; and moreover, in this case, when the negative shock hits, entrepreneurs default badly, unable to make any payment for the interest of their loans (while fully repaying the principal). Consequently, the nominal profit margin of bank lending is nil in the bad state. In this scenario, the following nominal intervention produces real effects. Whenever the negative productivity shock hits, the central bank of the economy prints fiat money to buy corn on the market. This fiat money then joins the sales revenue of entrepreneurs and subsequently flows to banks in loan repayments. Lastly, the central bank retires the injected fiat money and winds down the operation by demanding banks to exchange a quota of fiat money for its corn at the same market rate.

This intervention produces real effects via two channels. (1) By injecting money into the corn market, the intervention raises the corn price, thence the sales revenue of entrepreneurs, thence their loan repayments, and thence the nominal profit margin of bank lending in the bad state. In equilibrium, the average lending profit margin is a constant, determined by the condition that the engendered return to holding fiat money exactly offsets the cost of holding it. Therefore, if the bad-state margin rises, the good-state margin falls. In the good state, entrepreneurs do not default and hence the lending profit margin is equal to the lending rate. Altogether, by raising the bad-state margin, the intervention reduces the lending rate and thus the funding cost of entrereneurs. (2) By raising the nominal price in the bad state, the intervention increases the price fluctuation. This raises the funding cost of entrepreneurs when they default in the bad state. Intuition is as follows. Because entrepreneurs default in the bad state, they care about the cost of loan repayments only in the good state. Hence, when they use money to repay the loans, money is worth its good-state value. However, ex ante when they borrow money, its value is the average of the good-state value and the bad-state value. Therefore, with each unit of money entrepreneurs borrow, they obtain its average value, but pays back the good-state value. The ratio of the good-state value over the average value thus compounds their funding cost. The more the nominal price fluctuates, the more the money's real value fluctuates; hence, the greater the compounding factor and the higher the funding cost.

The two channels therefore produce conflicting effects. The net effect depends on how the intervention is wound down. It is wound down by demanding banks to exchange a quota of fiat money with the central bank. If the quota for a bank is proportional to its lending scale, then in net the intervention only raises entrepreneurs' funding cost and reduces efficiency. However, if that quota is fixed, independent of the lending scale, then the policy intervention raises efficiency and we characterise the optimal intervention scale.

This paper also explains why banks' money creation matters for efficiency. With money creation, one unit of fiat money enables the lending of $\chi > 1$ units of bank money We show that the bigger the money creation scale χ, the lower the lending rate. One might think that is because a bigger χ enlarges the supply of bank money and thus reduces the cost of borrowing it. However, this argument is flawed. Indeed, the supply is also enlarged if χ is fixed, but the quantity of fiat money rises, which, however, has no real impact whatsoever. The reason, according to this paper, is that money creation leverages up the return to holding fiat money, which is equal to χ times the profit margin of lending bank money. This return, in equilibrium, is equalised to the marginal cost of holding fiat money, a constant. Hence, the bigger is χ, the lower the equilibrium lending rate.

Literature

This paper joins the long theoretical discussion on the non-neutrality of nominal operations. Traditionally, the literature focuses on the role of fiat money as a means of payment for goods and services.[3] Recently several studies derive non-neutrality from the impact of the nominal operation for banks' money creation. Bianchi and Bigio (2017) underline the search friction on the interbank reserve market and show that a nominal policy produce real effects by altering the trade-off that banks face in allocating funds between assets of different levels of liquidity. Wang (2019) shows that a quantitative easing nominal operation enlarges the real money supply and improves efficiency

[3] This role is abstracted from in the New Keynesian literature, but is modelled in the Cash-In-Advance literature; see Walsh (2010) for a survey of both strands of literature. These strands of literature, in order to generate monetary non-neutrality, usually resort to nominal rigidity (e.g. menu costs or sticky portfolios), or incomplete information on monetary shocks (see the seminal work of Lucas 1972 and Angeletos and Lian 2016 for a survey), or exogenous rules on banks' holding of excess reserves (see Chen 2018 and Mishkin 2016). Fiat money's role as a means of payment is endogenised based a search friction in the literature following the seminal work of Kiyotaki and Wright (1989). This literature develops by Lagos and Wright (2005) into the New Monetarism; see Lagos et al (2017) and Rocheteau and Nosal (2017) for a survey and Lahcen (2019) for a development that combines labour search with money search.

by relaxing a real constraint that limits banks' capacity of lending bank money. Wang (2022) considers the collateral constraint on the interbank reserve bank and shows a change in the ratio of fiat money to a nominal bond can alter the tightness of this constraint, thereby producing real effects in the steady state. Both Bianchi and Bigio (2017) and Wang (2019) emphasize the role of fiat money in banks' liquidity management. This emphasis is shared by the present paper, where fiat money is necessary for banks to lend bank money, which, deep down, is because banks need fiat money to meet their liquidity demands.[4] To the long literature on nominal non-neutrality, the present paper makes two contributions. First, it considers nominal interventions on a product market, which have not been considered thus far. Second, the mechanism for nominal non-neutrality is new. In this paper, the nominal interventions produce real effects via their impacts on the nominal price fluctuation and the profit margin of bank lending. Neither of these channels has been considered.

This paper builds on a general equilibrium analysis of banks' money creation. Analysises of this kind are also to be found in a recent strand of literature that are not concerned with nominal non-neutrality; see, among others, Donaldson et al (2018), Jakab and Kumhof (2015), Kumhof and Wang (2018), Mendizábal (forthcoming), Morrison and Wang (2018).

The rest of the paper is organized as follows. Section 2 expounds that bank money is bank liability and banks create money by lending out their liabilities. Section 3 sets up the model, which is solved in Section 4. Nominal interventions on the product market are considered in Section 5. And Section 6 concludes.

2 The fact: Bank money is bank liability

Historically, bank money often takes the form of banknotes. They are a certificate of the bank's promise to pay a certain universally accepted means of payment, typically

[4]Unrelated to nominal non-neutrality, De Fiore et al (2018) and Piazzesi and Schneider (2018) also emphasize bank liquidity in their analysis of monetary policy, while Cavalcanti et. al. (1999) and Ennis (2018) among other examine banks' decision on reserve holding. On the other hand, this decision is unconsidered in studies where the reserve holding is pinned down by the binding reserve constraint; see e.g. Goodfriend and McCallum (2007).

gold in Europe and silver in China; a sample is illustrated below.[5]

Figure 1: A historical banknote, which reads "I promise to pay..." in the red box which is added by the author.

The money that a bank lends out, as is certified on its banknotes, is therefore the bank's promise to pay, namely its liability. Nowadays, as a rule, private banks do not print their notes. Bank money mainly takes the form of bank deposits and banks lend money by creating deposits, with double-ledger entries. For example, suppose the HSBC lends to a firm £10 million at an interest rate of 8%. Never does the bank hand the firm sacks of Bank of England notes for that purpose (how awkward!). What the bank does is: Credit £10 million into the firm's deposit account on the liability side and enter a loan of £10 $(1 + 8\%)$ on the asset side, that is, changes its balance sheet as follows.

Assets (in million pounds)	Liabilities (in million pounds)
Old Assets (X)	Old Liabilities (X)
Loan to the firm $(10\,(1+8\%))$	Deposit of the firm (10)
	Interest earned by the HSBC $(10 \times 8\%)$

Table 1: The double-ledger entries whereby the HSBC lends £10 million to the firm at 8%

The 10 million pounds in the firm's deposit account is created by the HSBC during and for the lending. The deposit is a certificate of the bank's promise to pay, as are the banknotes that it used to print. The only difference between a deposit and a banknote is that the promise to pay is recorded in a different way. Indeed, the reason that the bank can freely create a deposit of 10 million pounds is exactly that this money is its promise to pay. Any person or entity is free to making such a promise printing "I

[5] Even nowadays, on the notes of Bank of England is the phrase "I promise to pay" still printed, a trace of the long history when it was a private commercial bank.

promise to pay 10 million pounds" on a slip of paper, but the specialty of banks is that only their IOUs are widely accepted as a means of payment.[6]

That banks make lending by creating liability (with clicks of a mouse) does not mean they have an infinitely large lending capacity. For one thing, they are typically subject to a set of regulatory constraints, such as capital and reserves adequacy requirements. More importantly, a bank's promise to pay is accepted as a means of payment by the general public only if they believe that the bank will make good its promise when being asked to; that is, whenever they come to the bank demanding withdrawals from their accounts, the bank is able to and will meet the demands. Therefore, a necessary condition for a bank to lend out its liability as money is that the general public believe its liabilities are well backed by its assets. Lending to unworthy borrowers will certainly ruin this belief and thence the bank's lending business altogether.

3 The Model

The time $t = 0, 1, 2...\infty$. We will consider the steady state only and use the prime sign ' to denote the value in the next period. There is one consumption good, corn, which is perishable. The economy is populated by a continuum of $[0, 1]$ of banks, a continuum of $[0, 1] \times [0, 1]$ of entrepreneurs and many more workers. All of the agents are risk neutral and protected by limited liability. Banks live forever, with discount factor β. Entrepreneurs and workers live for one period. Workers either produce w units of corn in autarky or are employed by entrepreneurs. If an entrepreneur employs l workers at the beginning of the period, then he produces y units of corn at the end of the period, where

$$y = A_s z^{1-\alpha} l^{\alpha}.$$

Here $0 < \alpha < 1$ and z is the entrepreneur's huaman capital; without losing generality, we normalize $z = 1$. We will refer to l as the scale of the project. Productivity A_s depends on the state of the economy $s \in \{g, b\}$ which realises at the end of the period, where $A_g \geq A_b > 0$; hence g represents the good state, b the bad state. At the beginning

[6] As for why only banks have this privilege whereas others do not, see Kiyotaki and Moore (2001) and Wang (2019).

of each period, it is common knowledge that

$$s = \begin{cases} g, & \text{with probability } q > 0; \\ b, & \text{with probability } 1 - q > 0. \end{cases}$$

Let $A_e := qA_g + (1-q)A_b$ denote the mean and

$$\sigma := \frac{A_g}{A_b} \geq 1$$

represent the scale of the productivity fluctuation. The productivity shock s i.id over time. Some of our results are conditioned on q being close to 1. The interpretation is that the good state represents the "*normal times*", and the bad state represents a negative shock, which happens only infrequently.

To hire workers, entrepreneurs have to borrow bank money as a means of wage payment. As explained in Section 2, in reality, this money is bank liability; the money lent out by one bank is the bank's promise to pay fiat money. In the model economy, we assume there are H units of fiat money, and *one unit of bank money is the issuing bank's liability to pay one unit of fiat money.* Let d denote the quantity of bank money that a representative bank lends out, and D the aggregate lending of bank money. In equilibrium, $D = d$ because there is a unit mass of banks.

As said in the Introduction, we underline not fiat money's role as a means of payment, but its role in bank lending. This role is modelled with the following *scale constraint*:

$$d \leq \chi h, \tag{1}$$

where h is the representative bank's fiat money holding and $\chi \geq 1$ is a constant. The scale constraint captures a variety of real life constraints to which banks are subject. First, the fiat money position h represents the bank's holding of reserves in reality and Constraint (1) thus represents the *reserve constraint* that banks' reserve rate h/d be no smaller than threshold $1/\chi$. Second, if we allow h to represent more broadly, banks' liquid-asset holding in reality, then Constraint (1) represents a certainly *liquidity constraint* studied by Wang (2022), which anchors a bank's lending scale d to its liquid asset position h. Lastly, because fiat money is the sole saving asset in the model economy, the fiat money position h also respresents the bank's wealth in reality. In this case

Constraint (1) represents a *leverage constraint* that the leverage rate d/h is no larger than χ. Observe that the special case in which $\chi = 1$ represents 100 percent reserved banking, where banks do not create money. In general, $\chi - 1$ measures the scale of banks' money creation.

The lending rate r is determined by the competitive market. If an entrepreneur borrows M units of bank money at the beginning of the peirod, then she is obligated to repay $M(1+r)$ units of bank money to her lender at the end of the peirod. The profit that banks obtain from lending is thus nominal, as in reality. Hence, in the model economy, as in reality, banks pay dividends with bank money to their shareholders, who then exchange it for corn on a competitive market. On the same market, workers employed by entrepreneurs use their wage incomes, which are paid with bank money, to buy corn. This market opens after the state $s \in \{g, b\}$ of the economy realises and entrepreneurs produce corn. On this market, then, entrepreneurs supply corn; banks and entrepreneur-employed workers supply bank money. Let p_s denote the quantity of corn that one unit of bank money is exchanged for in state s; that is, $1/p_s$ is the nominal price of corn.

When the market opens, entrepreneurs are going to exit the economy; they want bank money only to settle the loans. Their aggregate demand of bank money is hence $D(1+r)$ *if they do not default*, in which case bank lending earns a nominal profit margin of r. However, entrepreneurs might default (due to the negative productivity shock). In general, let γ_s denote the nominal profit margin of bank lending in state s. Then, $\gamma_s \leq r$ and entrepreneurs' aggregate demand of bank money – the total quantity of bank money that they manage to gather – is $D(1+\gamma_s)$ in state s. On the supply side of money, entrepreneur-employed workers are going to quite the economy also, so they spend all their wage incomes on corn. The quantity of bank money that they supply is equal to the aggregate wage income, which is D, because all the bank money that is lent out at the beginning of the period is used for hiring workers and ends up as their wage incomes. The quantity of bank money that banks supply to the corn market is equal to the aggregate nominal dividend. Let it be V_s in state s. Then, the aggregate supply of bank money on the market is $D + V_s$. The market clearing commands that

$D+V_s = D(1+\gamma_s)$, which leads to:

$$V_s = \underbrace{D\gamma_s}_{\text{Aggregate lending profit}}. \tag{2}$$

Therefore, in the model economy, the aggregate dividend that banks pay out is equal to the aggregate banks' lending profit. That is intuitive. In the model economy, the only saving asset is fiat money and fiat money is in a fixed supply; therefore, all the profit is paid out as dividends in equilibrium.

When selling corn, one entrepreneur might receive bank money issued by other banks than his lenders. Hence, when entrepreneurs use bank money to settle loans with their lenders, one bank's money might flow into another; say 10 units of Bank 1's money flow to Bank 2 and inversely 8 units of Bank 2's money flow into Bank 1. Recall that the a bank's money is the bank's promise to pay fiat money, namely, the bank's IOU. Therefore, the following inter-flows of banks' IOU occur in the example:

$$\text{Bank 1} \quad \underset{\text{IOU 8 units of fiat money}}{\overset{\text{IOU 10 units of fiat money}}{\rightleftarrows}} \quad \text{Bank 2.}$$

As a result, Bank 1's owes ten units of fiat money to Bank 2, Bank 2 eight units to Bank 1. All the interbank liabilities so formed are first netted and then cleared with fiat money; in the example above, the liabilities between the two banks are cleared with Bank 1 paying 2 units of fiat money to Bank 2. Consider now the representative bank. It has issued and lent out d_t units of bank money at the beginning of period t. Suppose later it issues v_s units of bank money to pay the dividend in state s. In total, therefore, $d+v_s$ units of the bank's money, that is, its liability, flow out into circulation. With the lending profit margin being γ_s, $d(1+\gamma_s)$ units of bank money flows into to this bank when its loans are settled. Its net interbank liability position is thus $(d+v_s)-d(1+\gamma_s) = v_s - d\gamma_s$, which the bank clears with fiat money. The bank's next-period fiat money position is hence $h'_s = h-(v_s-d\gamma_s)$, which can be rearranged into the following budget constraint.

$$d\gamma_s = v_s + (h'_s - h). \tag{3}$$

Namely, the bank's net lending profit of $d\gamma_s$ is either spent on dividend v_s, or saved to enlarge its fiat-money position by $h'_s - h$. A bank can freely create bank money to pay dividend, but this payout is not free: Its bank money is a liability that it needs to *fully* redeem with fiat money. If the bank pays out one more unit of bank money so that its shareholders have p_s more units of corn, then its future fiat-money position has to be one unit less. For banks, therefore, the exchange is between corn and fiat money and p_s is the price of fiat money in the unit of corn.

The timing of events at any period t is as follows.

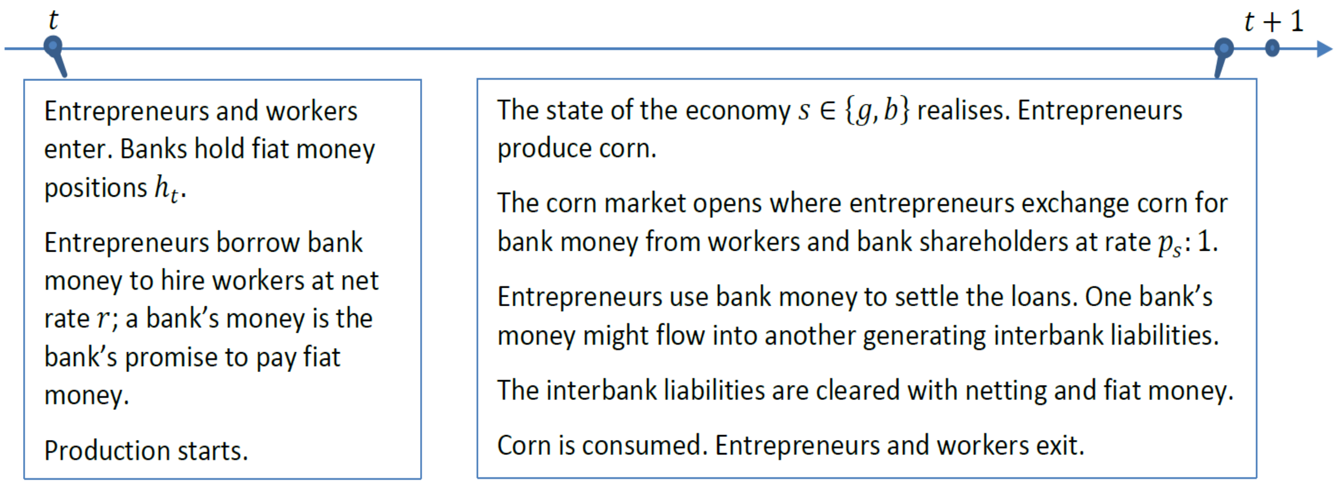

Figure 2: The timing of events in period t.

Passing on to the analysis of the market equilibrium, we examine the social planner's allocation as the benchmark, which concerns the number l of workers that each entrepreneur employs. Considering that the opportunity cost of a worker in this employment is w, the social planner's problem is

$$\max_{l} A_e l^a - wl.$$

Hence, in the first-best allocation, each entrepreneur hires l^{SB} workers, where

$$l^{SB} = \left(\frac{A_e \alpha}{w}\right)^{\frac{1}{1-\alpha}}. \tag{4}$$

4 The steady state of market equilibrum

In the steady state, a unit of bank money is of real value p_s in state $s \in \{g, b\}$ at the end of each period. Ex ante, when entrepreneurs use it to hire workers, it is worth p, where

$$p = qp_g + (1-q)\,p_b.$$

We define the *inverse price fluctuation*

$$\tau := \frac{p_b}{p_g}.$$

Then $\tau \leq 1$ and if $\tau < 1$, the nominal price fluctuates. Indeed, $\tau^{-1} = p_b^{-1}/p_g^{-1}$ measures the scale of the price fluctuation; however, it is mathematically convenient to use τ rather than τ^{-1}, as we will see. According to the Quantity Theory of Money, the unit value of money is proportional to the real output and hence to the realized productivity: $p_g/A_g = p_b/A_b$, or equivalently,

$$\tau = \sigma^{-1}. \tag{5}$$

It follows that $\tau < 1$ so long as $\sigma > 1$; that is, the nominal price fluctuates whenever the productivity fluctuates. That, we will see, is not the case in the model.

We begin with entrepreneurs' problem, then banks' problem, and lastly the market clearing.

4.1 The demand of bank money: Entrepreneurs' problem

Suppose an entrepreneur borrows M units of bank money at the beginning of the period. Their real value is Mp and enables her to hire $l = Mp/w$ workers. At the end of the period, given the lending rate r, she is obligated to repay $M(1+r)$ units of bank money, of which the real value is $M(1+r)p_s$ in state $s \in \{g, b\}$. If she cannot make the full repayment, she defaults and obtains zero profit. Hence, the entrepreneur's demand M of bank money solves the following problem.

$$\max_M \mathbf{E}_s \left[\max\left(A_s l^\alpha - M(1+r)p_s, 0\right)\right]$$
$$\text{s.t. } l = \frac{Mp}{w}.$$

Entrepreneurs will not default in the good state at the optimum; otherwise, they would default in both states and the factor of production that they contribute, i.e. their human capital $z = 1$, would earn zero return. However, only the possibility of default in the bad state needs be considered.

Let

$$\theta(\sigma) := \frac{q}{1-q}\left[\left(1+\frac{1-q}{q}\sigma^{-1}\right)^{\frac{1}{\alpha}} - 1\right]. \tag{6}$$

Then $\theta'(\sigma) < 0$ and $\theta(\sigma) > (\alpha\sigma)^{-1}$.[7]

Lemma 1 *If $\tau < \theta(\sigma)$, entrepreneurs do not default in the bad state and their project scale $l = l^{nd}$, where*

$$l^{nd} = \left(\frac{A_e\alpha}{w(1+r)}\right)^{\frac{1}{1-\alpha}}. \tag{7}$$

If $\tau > \theta(\sigma)$, then entrepreneurs default in the bad state and $l = l^d$, where

$$l^d = \left(\frac{A_g\alpha}{w(1+r)\times p_g/p}\right)^{\frac{1}{1-\alpha}}. \tag{8}$$

If $\tau = \theta(\sigma)$, then entrepreneurs are indifferent between the default scale l^d and the no-default scale l^{nd}, and $l^d > l^{nd}$. In all the cases, their demand of bank money is

$$M = \frac{wl}{p}.$$

Proof. See Appendix. ■

By the lemma, the optimal scale of entrepreneurs' project falls into two regimes, one in which entrepreneurs default in the bad state, the other in which they do not; and which regime they select into depends on whether $\tau < \theta(\sigma)$. Intuitively, because the loans are nominal, entrepreneurs default in the bad state if and only if the nominal revenue in the state is sufficiently lower than that in the good state. The former is in proportion to $A_b \cdot (p_b)^{-1}$, the latter to $A_g \cdot (p_g)^{-1}$. It follows that the default regime prevails if $A_b \cdot (p_b)^{-1}$ is sufficiently low relative to $A_g \cdot (p_g)^{-1}$, or equivalently τ is sufficiently high relative to σ^{-1}, of which the exact condition is $\tau > \theta(\sigma)$ according to the lemma. Indeed, if the Quantity Theory of Money, and thus equation (5), holds true, then $\tau = \sigma^{-1} < \alpha^{-1}\sigma^{-1} < \theta(\sigma)$; accordingly entrepreneurs never default, no matter how low A_b is relative to A_g. This is Intuitive: By the Quantity Theory of Money, entrepreneurs' nominal revenue is independent of state s, so that they are solvent in the bad state as much as they are in the good state.

If $\tau = \theta(\sigma)$, both the default scale l^d and no-default scale l^{nd} are optimal for entrepreneurs. In this case, $l^d > l^{nd}$. Intuitively, that is because if an entrepreneur

[7] That is because $(1+x)^y > 1 + xy$ for $x > 0$ and $y > 1$.

anticipates no default, he cares about the output of his project in both states and the marginal product of labor for him is thus proportional to A_e. However, if he anticipates default in the bad state, then he cares only about the output in the good state and thus *for him*, the marginal product of labor is proportional to A_g. That $A_g > A_e$ drives that $l^d > l^{nd}$.

It is convenient to normalize the equilibrium project scale by the first-best one l^{FB}. Define the *normalized project scale* e as follows.

$$e := \left(\frac{l}{l^{FB}}\right)^{1-\alpha}.$$

This e is also an *efficiency index*. The first-best efficiency is normalised to 1. Any deviation from $e = 1$ means an efficiency loss. If $e < 1$, entrepreneurs' projects are too small; and if $e > 1$, they are too big. With l^{FB} given (4) and l by Lemma 1, unless $\tau = \theta(\sigma)$, the normalized project scale e is a function of (τ, r) as follows.

$$e(\tau, r) = \begin{cases} \frac{1}{1+r}, & \text{if } \tau < \theta(\sigma); \\ \frac{1}{q+(1-q)\sigma^{-1}} \frac{q+(1-q)\tau}{1+r}, & \text{if } \tau > \theta(\sigma). \end{cases} \tag{9}$$

Observe that the normalized scale always decrease with the lending rate r. Intuitively, the higher the lending rate, the more expensive the loans; therefore the less entrepreneurs borrow and the smaller their projects. If $\tau < \theta(\sigma)$ and thus the no-default regime prevails, the lending rate r is all that matters for the project scale, and thus for efficiency, by (9). However, if $\tau > \theta(\sigma)$ and thus the default-regime prevails, the project scale also decreases with the price fluctuation τ^{-1}. Intuition is as follows. In the default regime, entrepreneurs will default in the bad state and they care about the cost of loan repayment only in the good state. Therefore, a unit of bank money costs p_g when they repay loans. However, it is worth p when they borrow it. That is, with each unit of bank money borrowed, entrepreneurs pay p_g for something worth p. The rate p_g/p, which is equal to $(q + (1-q)\tau)^{-1}$, compounds their borrowing cost. For them, hence, the effective gross borrowing rate is $(q + (1-q)\tau)^{-1}(1+r)$ (rather than $1+r$), which explains why in (9) the normalized scale is proportional to the inverse of this term in the default regime. The larger the price fluctuation τ^{-1}, the greater the compounding factor, the higher the effective borrowing cost, and the smaller the project scale.

4.2 The supply of bank money: Banks' problem

We begin with calculating the nominal profit margin γ_s of bank lending in state $s \in \{g, b\}$. The profit margin is equal to the lending rate r if entrepreneurs do not default, which is the case if $s = g$, or if $s = b$ and $\tau < \theta(\sigma)$ by Lemma 1. If they default, then all their nominal revenue $A_b \left(l^d\right)^\alpha / p_b$ is used to repay the loan of $M = wl^d/p$ units of bank money. Hence, the bad-state nominal profit margin in the default regime is:

$$\gamma_b^d = \frac{A_b \left(l^d\right)^\alpha / p_b}{w \left(l^d\right)^\alpha / p} - 1|_{(8)} = \frac{1+r}{\sigma\alpha\tau} - 1. \tag{10}$$

The default regime arises only if $\tau \geq \theta(\sigma)$, in which case $\sigma\alpha\tau > 1$ because $\theta(\sigma) > (\alpha\sigma)^{-1}$. Hence, the profit margin is indeed lower when entrepreneurs default:

$$\gamma_b^d < r. \tag{11}$$

Altogether, the nominal profit margin of bank lending is a function of the market lending rate r and the realized state s as follows.

$$\gamma_s(r) = \begin{cases} r, & \text{if } s = g; \text{ or } s = b \text{ and } \tau < \theta(\sigma); \\ \frac{1+r}{\sigma\alpha\tau} - 1, & \text{if } s = b \text{ and } \tau > \theta(\sigma). \end{cases} \tag{12}$$

If the representative bank pays out v_s units of bank money as the dividend, the real value of the dividend is $v_s p_s$. The decision problem of thebank is:

$$V(h) = \max_{d, \{v_s, h'_s\}_{s\in\{g,b\}}} \mathbf{E}\left(v_s p_s + \beta V(h'_s)\right), \tag{13}$$

$$\text{s.t. } v_s = h + \gamma_s(r) d - h'_s; \tag{14}$$

$$v_s \geq 0; \tag{15}$$

$$d \leq \chi h.$$

Here Equation (14) follows from Equation (3) and $d \leq \chi h$ is due to the scale constrait (1). A negative dividend v_s means that the bank calls its shareholders to make new contribution of capital. With Constraint (15), we assume that this measure is prohibitively costly to banks. This constraint is thus referred to as the *no-call* constraint.

The no-call constraint in the good state, $v_g \geq 0$, will never bind. That is because by Equation (2), the aggregate dividend is equal to the aggregate bank profit; and hence no dividend in the good state implies a zero good-state profit margin, i.e. $r = 0$. Because the bad-state profit margin of bank lending $\gamma_b \leq r$, it is also nil then. it cannot be the case that bank lending always earns zero retun in equilibrium. Intuitively, to make lending (of bank money), banks need to hold fiat money, which is costly due to time preference. To provide banks with incentives to hold fiat money, bank lending must earn a positive rate of return. Indeed, define

$$r_f := \frac{1}{\chi}\left(\frac{1}{\beta} - 1\right) \tag{16}$$

as *the required risk-free lending rate*. The idea is that due to money creation, holding one unit of fiat money allows the bank to lend out χ units of bank money, each of which earns return at rate r if entrepreneurs never default. Consequently, the rate of return to holding fiat money is χr. To have this return rate equalized to the cost of holding fiat money, which is $1/\beta - 1$, the requred lending rate is r_f.

Let $(1 - q)\,\mu$ be the Lagrangean multiplier for the bad-state no-call constraint $v_b \geq 0$. Then μ represents the tightness of the no-call constraint conditional on $s = b$.

Lemma 2 *The solution to the representative bank's problem (13) satisfies the following claims.*

(i) Either $\tau = 1$ or $\mu > 0$.

(ii)

$$\mathbf{E}_s\left(\gamma_s\left(r\right)\right) = r_f. \tag{17}$$

(iii) The scale constraint (1) is binding: $d = \chi h$.

Proof. See Appendix. ■

Claim (i) follows from

$$p_g = p_b\left(1 + \mu\right). \tag{18}$$

Intuitively, at the end of each period, the value of a unit of fiat money is equal to the sum of the discounted cash flows that it will generate in the future. As the future states of the economy are independent of its present state, so are the future cash flows, and so should be the real value of fiat money. If fiat money is worth less in the bad state than in the good state – i.e. $p_b < p_g$ – then the difference must be made up by the shadow value μ of fiat money in relaxing the no-call constraint $v_b \geq 0$. Hence Equation (18). It follows that either $p_b = p_g$, i.e. $\tau = 1$; or $\mu > 0$ and the constraint $v_b \geq 0$ is binding, that is,

$$h + \gamma_b(r)d - h_b' = 0. \tag{19}$$

Equation (17) is driven by Equation (18), according to which if the shadow value of bank money is taken into account, the "real" value of money is a constant, independent of s. Hence, the expected real return rate of bank lending is equal to the expected nominal return rate, $\mathbf{E}_s(\gamma_s(r))$. In equilibrium, the expected real return rate must be equalised to the required risk-free lending rate r_f, in order to provide banks with incentives to hold fiat money. Hence, Equation (17).

The last claim is driven by the same requirement that in equilibrium, banks need obtain a positive benefit from holding fiat money. At the time of lending, the benefit consists in relaxing the scale constraint. Should the constraint be non-binding, the benefit would be nil, which violates the equilibrium requirement.

4.3 Market clearing and definition of steady state

In the steady state,

$$h = h_s' = H, \text{ for } s \in \{g, b\}. \tag{20}$$

With this equaiton, the binding bad-state no-call constraint (19) leads to

$$\gamma_b(r) = 0. \tag{21}$$

By Claim (i) of Lemma 2, hence, either $\tau = 1$ or $\gamma_b(r) = 0$. That is, *the nominal price fluctuates if and only if bank lending earns nil nominal profit margin in the bad state*. This is due to a general equilibrium effect. Intuitively, if $\tau < 1$, in the bad state, fiat money is too cheap and banks wants to exchange corn for fiat money as much as possible. Due to the no-call constraint $v_b \geq 0$, they cannot obtain extra corn from their shareholders for this purpose. The best they can do, therefore, is abstain from issuing new bank money for dividend payout (see Equation 3). As a result, on the corn market only entrepreneur-employed workers supply bank money. Their supply is in quantity D, the quantity that entrepreneurs have borrowed in the first place. Consequently, the money entrepreneurs obtain is exactly sufficient for them to pay the principals of their loans and bank lending earns a zero nominal profit margin. Observe that as the workers always supply D units of bank money on the corn market, so can entrepreneurs always repay the principals of their loans in full.

Now consider the loan market at the beginning of each period. Because $h = H$ in the steady state, by Claim (iii) of Lemma 2, the supply of bank money is

$$D = \chi H. \tag{22}$$

On the demand side, entrepreneurs' demand for bank money is $M = wl/p$ and $l = l^{FB} e(\tau, r)^{\frac{1}{1-\alpha}}$, where function $e(\tau, r)$ is given by (9). The market clearing commands $M = D$, namely:

$$\frac{wl^{FB} e(\tau, r)^{\frac{1}{1-\alpha}}}{p} = \chi H. \tag{23}$$

Due to lemma 2, we can define steady-state equilibrium as follows.

Definition 1 *A profile $\{\tau^*, r^*, p^*\}$ forms a steady state if and only if it satisfies the following conditions.*

(1) Either $\tau^* = 1$ or $\gamma_b(r^*) = 0$.

(2) $\mathbf{E}_s(\gamma_s(r^*)) = r_f$.

(3) The loan market clears: Equation (23) holds.

Conditions (1) and (2) are independent of the quantity H of fiat money. Hence so are τ^*, r^* and the efficiency index $e(\tau^*, r^*)$. Only condition (3) is affected by H: From (23),

$$\frac{1}{p^*} = \frac{\chi}{wl^{FB} e(\tau^*, r^*)^{\frac{1}{1-\alpha}}} \times H. \tag{24}$$

It follows that the quantity of fiat money has an impact only on the *ex ante* nominal price level $(p^*)^{-1}$ in the steady state. Moreover, $d\ln(p^*)^{-1}/d\ln H = 1$, that is, the percentage change to the ex ante nominal price is equal to the percentage change to the quantity of fiat money. Both are predictions of the Quantity Theory of Money. The real efficiency e depends on (τ^*, r^*) only, which is what we track below.

4.4 The three phases of the steady state

The steady state, we will show, is unque except at a knife-edge case. With $\sigma = A_g/A_b$ increasing from 1, this unique steady state proceeds through three phases. For intuition, take A_g as given and let A_b gradually fall from A_g. To begin with, if $A_b \lessapprox A_g$, the difference between the good and bad states is small and entrepreneurs do not default in the bad state (as they do not in the good state). By Lemma 1, hence, Phase 1 will be characterized by

$$\tau^* < \theta(\sigma). \tag{25}$$

In this Phase, $\gamma_b = r$ and, by (17), $r = r_f > 0$. It follows from equilibrium Condition (1) that $\tau^* = 1$. Now consider A_b keeps falling in this phase. As the nominal price in the bad state is fixed at the good-state level and is not rising, entrepreneurs' nominal revenue falls with A_b, and so does their ability to fully repay their loans – the principal and the interest. Hence, if A_b falls to a threshold A_{b1}, entrepreneurs will default in the bad state. Below the threshold, the steady state enters Phase 2.

In Phase 2, if $A_b \lessapprox A_{b1}$, though in the bad state, entrepreneurs are unable to make the full repayment of the interest of their loans, the shortfall will not be large. Namely, entrepreneurs will be able to pay part of the interest in the bad state (recall that they are always able to pay the principal). As a result, bank lending still earns a positive profit margin in the bad state. Hence, Phase 2 is characterized by

$$\tau^* > \theta(\sigma) \wedge \gamma_b > 0. \tag{26}$$

As $\gamma_b > 0$ still, so $\tau^* = 1$. In Phase 2, as in Phase 1, with A_b falling, the bad-state nominal price is still not rising to offset the damaging effect of the fall in A_b for entrepreneurs' nominal revenue. Hence, if A_b falls to an even lower threshold A_{b2}, entrepreneurs are unable to make any interest payment at all and the bad-state profit margin is completely annihilated.

If A_b falls below A_{b2}, the steady state enters Phase 3. Hence, Phase 3 is characterized by

$$\tau^* > \theta(\sigma) \wedge \gamma_b = 0. \tag{27}$$

In this phase, the nominal price fluctuates and bank lending earn nil profit margin in the bad state.

The intuition stated above leads to Proposition 1 below.

Let:

$$\sigma_{c1} \quad : \quad = \frac{\frac{1}{q}-1}{\left(\frac{1}{q}\right)^{\alpha}-1}; \tag{28}$$

$$\sigma_{c2} \quad : \quad = \frac{1}{\alpha}\frac{\frac{1}{\beta}-1+\chi q}{\chi q}. \tag{29}$$

Then, $\sigma_{c1} > \alpha^{-1}$.[8] We assume:

Assumption 1: $\sigma_{c2} > \sigma_{c1}$, or equivalently,

$$\chi < \left(\frac{1}{\beta}-1\right)\frac{\left(\frac{1}{q}\right)^{\alpha}-1}{q\left[\alpha\left(\frac{1}{q}-1\right)-\left(\left(\frac{1}{q}\right)^{\alpha}-1\right)\right]}. \tag{30}$$

The right hand of Inequality (30) is positive[9] and goes to infinity if $q \to 1$.

Proposition 1 *Under Assumption 1, the steady state is as follows.*

1. *If $\sigma \in [1, \sigma_{c1}]$, the steady state is in Phase 1, where Condition (25) holds, so entrepreneurs never default. In this phase,*

$$\begin{aligned} \tau^* &= 1 \\ r^* &= r_f. \end{aligned}$$

2. *If $\sigma \in [\sigma_{c1}, \sigma_{c2}]$, the steady state is in Phase 2, where Condition (26) holds, so in $s = b$, entrepreneurs default, but make a partial interest payment. In this phase,*

$$\begin{aligned} \tau^* &= 1 \\ r^* &= \frac{r_f + (1-q)\left(1-(\sigma\alpha)^{-1}\right)}{q+(1-q)(\sigma\alpha)^{-1}}. \end{aligned} \tag{31}$$

3. *If $\sigma \geq \sigma_{c2}$, the steadystate is in Phase 3, where Condition (27) holds, so entrepreneurs make no interest payment in $s = b$. In this phase,*

$$\tau^* = \frac{r_f + q}{q\alpha\sigma} \tag{32}$$

$$r^* = \frac{1}{q}r_f. \tag{33}$$

[8] That is because if $x > 1$ and $\alpha \in (0,1)$, then $f(x) := \alpha(x-1) - (x^{\alpha}-1) > 0$ as $f(1) = 0$ and $f'(x) = \alpha\left(1-x^{\alpha-1}\right) > 0$.

[9] $\alpha\left(\frac{1}{q}-1\right) - \left(\left(\frac{1}{q}\right)^{\alpha}-1\right) > 0 \Leftrightarrow \sigma_{c1} > \frac{1}{\alpha}$.

The normalized project scale in the steady state depends on σ as follows.

$$e^*(\sigma) = \begin{cases} \frac{1}{1+r_f}, & \text{if } \sigma < \sigma_{c1}; \\ \frac{1}{1+r_f} \cdot \eta(\tau^*(\sigma), \sigma), & \text{if } \sigma > \sigma_{c1}, \end{cases} \tag{34}$$

where

$$\eta(\tau, \sigma) := \frac{\left[q + (1-q)(\sigma\alpha)^{-1}\tau^{-1}\right]\left[q + (1-q)\tau\right]}{q + (1-q)\sigma^{-1}}. \tag{35}$$

$e^{*\prime}(\sigma) < 0$ *if* $\sigma > \sigma_{c1}$, $\lim_{\sigma\to\infty} e^*(\sigma) = \frac{q}{1+r_f}$, *and* $e^*(\sigma_e) = \frac{1}{1+r_f}$ *for some* $\sigma_e > \sigma_{c2}$.

Proof. See Appendix. ■

Functions $(r^*(\sigma), \tau^*(\sigma))$ are illustrated in the figure below.

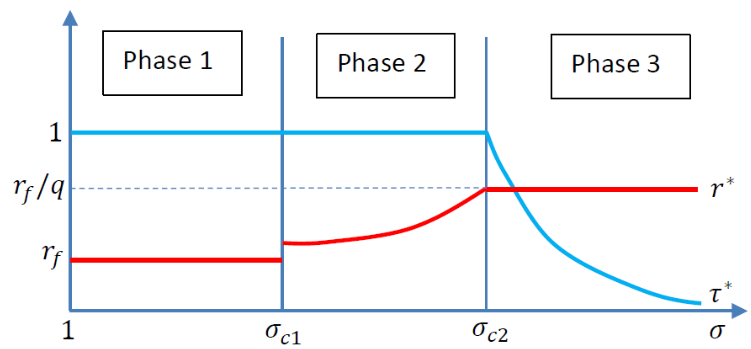

Figure 3: The diagrams of $\tau^*(\sigma)$ (the blue line) and $r^*(\sigma)$ (the red line).

Function $e^*(\sigma)$ is illustrated below.

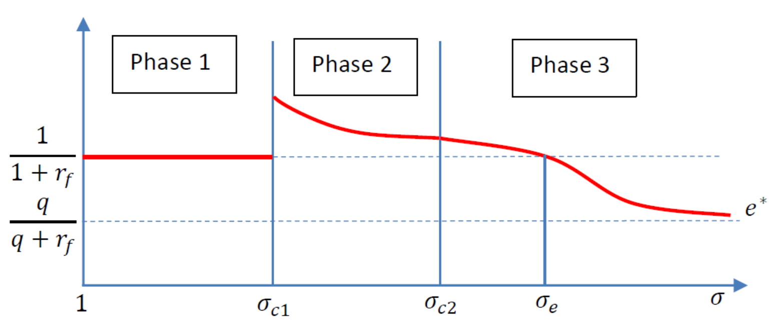

Figure 4: The diagram of $e^*(\sigma)$.

We make some observations regarding the proposition.

First, from Figure 3, the price start fluctuating – $\tau^* < 1$ – only when the productivity fluctuation is large enough, above σ_{c2}. That is driven by the result that $\tau^* < 1$ if and only if the bad-state profit margin of lending is nil, which holds only if A_b is low enough relative to A_g, namely, $\sigma = A_g/A_b$ is high enough.

Second, from Figures 3 and 4, both the lending rate r^* and the normalized project scale e^* are discontinuous at $\sigma = \sigma_{c1}$. Actually $\sigma = \sigma_{c1}$ is a knift-edge case in which $\tau^* = \theta(\sigma) = 1$ and entrepreneurs are indifferent between the default and no-default scales. In this case, there is a continuum of equilibria, each characterized by a fraction $\xi \in [0, 1]$ of entrepreneurs who choose the default scale. If $\xi = 0$, the equilibrium is in Phase 1 where no entrepreneurs default; if $\xi = 1$, the equilibrium is in Phase 2 where all entrepreneurs default in $s = b$.

Third, from Figure 3, the lending rate r^* surges when σ ascends above threshold σ_{c1},[10] and increases with σ throughout Phase 2, and then stays constant in Phase 3. All these results are driven by Equation (17), which commands that the expected nominal profit margin is equal to r_f, whereby

$$r = \frac{r_f - (1-q)\gamma_b}{q}. \tag{36}$$

The rate r surges at $\sigma = \sigma_{c1}$ because γ_b discontinuously falls at the point: $\gamma_b = r$ in Phase 1, but in Phase 2 where entrepreneurs default, $\gamma_b = \gamma_b^d < r$ by (11). Also, as γ_b decreases with σ throughout Phase 2 (as intuitively argued above), so r increases with it. Finally, in Phase 3, $\gamma_b = 0$ throughout so that $r = r_f/q$ is a constant.

Fourth, from Figure 4, the normalized project scale $e^* = \left(l^*/l^{FB}\right)^{1-\alpha}$ surges when σ ascends above threshold σ_{c1}. This occurs because at $\sigma = \sigma_{c1}$ the equilibrium flips from the no-default regime to the default regime and hence the project scale surges from l^{nd} to l^d. Moreover, e^* keeps decreasing in both Phases 2 and 3 (i.e. $e^{*\prime}(\sigma) < 0$ if $\sigma > \sigma_{c1}$). That is because in both Phases the equilibrium is in the default regime, where, by the discussion of Equation (9), the project scale decreases with both the lending rate and the price fluctuation. In Phase 2, the lending rate r^* increases with σ, while the price fluctuation $\tau^{*-1} = 1$ is fixed. In Phase 3, the price fluctuation τ^{*-1} increases with σ while the lending rate $r^* = r_f/q$ is fixed. Therefore, e^* decreases with σ in both Phases. Indeed, despite the surge at $\sigma = \sigma_{c1}$, it falls back to the level in the no-default regime – i.e. $1/(1 + r_f)$ – at $\sigma = \sigma_e$.

Fifth, by (34) the normalized project scale e^* always decreases with $r_f = \left(\frac{1}{\beta} - 1\right)/\chi$ and therefore increases with χ. Hence,

[10] Formally, $r^*(\sigma_{c1}) > \frac{1}{1+r_f} \Leftrightarrow \sigma_{c1}\alpha > 1 \Leftrightarrow \sigma_{c1} > \frac{1}{\alpha}$.

Corollary 1 *The larger the scale χ of banks' money creation, the bigger the projects of entrepreneurs.*

This real implication of banks' money creation function is driven by the effect that the larger the scale of money creation, the cheaper the borrowing of bank money (i.e. $\partial r^*/\partial x < 0$). Intuion is as follows. Recall from the discussion of Equation (17) that the equilibrium lending rate r^* is determined by the requirement that the return rate generated by bank lending for holding fiat money be equalized to the cost of holding, $(1-\beta)/\beta$. Money creation levers up this return rate: Holding one unit of fiat money enables lending of χ units of bank money and the return rate of holding fiat money is thus χ times of that of bank lending. The larger the scale χ of money creation, the lower the return rate of bank lending that is needed to generate a rate of return $(1-\beta)/\beta$ to holding fiat money.

We have seen that the entrepreneurs' project is too small relative to the first-best size – i.e. $e^* < 1$ – throughout Phase 1. However, e^* surges at $\sigma = \sigma_{c1}$. In certain quarters of the parameter space, this surge is so large that in Phase 2, $e^* > 1$ and ineffiency is in the opposite direction: The projects are too big and bank loans are too cheap. This is probably a scenario less relevant. Therefore, we make the following assumption, which ensures that entrepreneurs' projects are always too small.

Assumption 2:

$$\chi < \frac{\frac{1}{\beta} - 1}{(1-q^{\alpha})\left(\frac{1}{\alpha} - 1\right)}. \tag{37}$$

Inequality (37) holds if q is sufficiently close to 1 because its right hand side goes to infinity if $q \to 1$.

Lemma 3 *Under Assumption 2, $e^* < 1$ for all σ. This assumption is stronger than Assumption 1.*

Proof. See Appendix. ■

Given Assumption 2 is stronger than Assumption 1, only the former is made throughout the paper. With Assumption 2, the higher the normalized project scale e^*, the more efficient the equilibrium. Hence, *the normalized project scale measures the efficiency level.*

5 Nominal interventions on the corn market

Consider Phase 3 of the steady state. In this phase, entrepreneurs default so badly in the bad state that they are unable to make any interest payment, so the bad-state lending profit margin is nil; as a result, banks charge the highest interest rate, causing entrepreneurs' projects to be severely under-sized. It seems a nominal intervention that increases the bad-state profit margin can help. Consider the following one. *Whenever the bad state realises*, first, the central bank prints Φ units of fiat money and uses them to buy corn on the market. One unit of fiat money will have the same real value p_b as one unit of bank money, because the latter is just a liability to pay the former and will be fully redeemed with it at the clearing stage; the two types of money are thus equivalent in value. With Φ units of fiat money, therefore, the central bank buys Φp_b units of corn from entrepreneurs. In loan repayments, the Φ units of fiat money, along with bank money, flow to banks. Then, after the interbank liabilities are cleared, the central bank winds down its intervention by demanding banks to swap a quota of their fiat money for the central bank's corn at the market rate of $1 : p_b$. The quota can vary across banks.

The timing of events with the nominal intervention is thus illustrated in Figure 5 below.

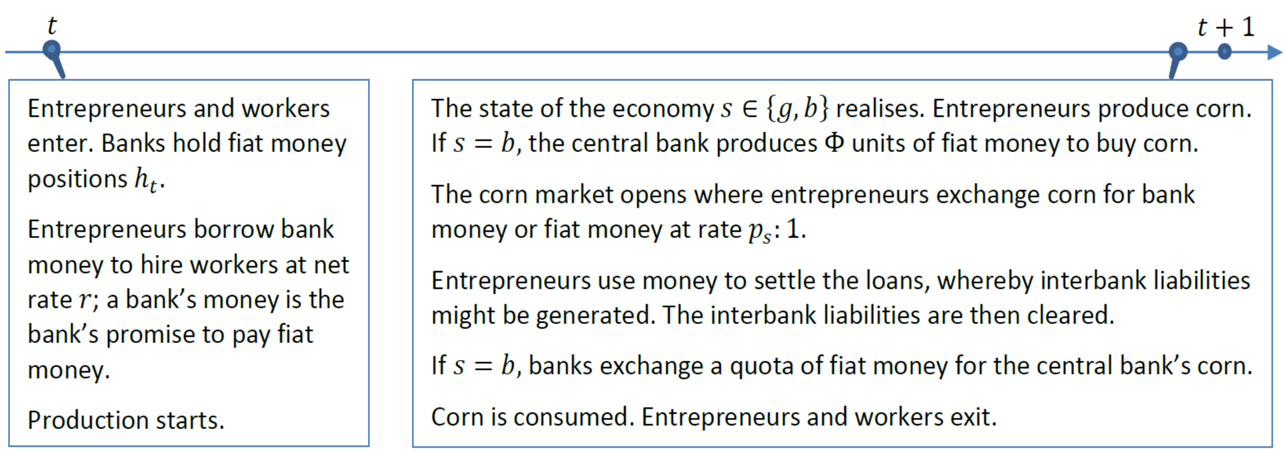

Figure 5: The timing of events in period t with the nominal intervention.

The quota φ for an individual bank can be set in two ways, giving rise to two winding-down mechanisms and accordingly, two policies.

Policy 1: φ is proportional to the bank's lending scale d, that is, $\varphi = \phi d$ for some ϕ. To retire all the injected fiat money,

$$\phi = \frac{\Phi}{D(\Phi)}, \tag{38}$$

where $D(\Phi)$ is the aggregate bank lending conditional on the intervention of scale Φ.

Policy 2: φ is a fixed quantity, independent of the bank's lending scale. To retire all the injected fiat money, φ satisfies

$$\int_0^1 \varphi di = \Phi.$$

In both scenarios, it is more convenient to use the ratio $\phi = \Phi/D$ instead of Φ to represent the scale of the policy intervention. As individual banks take both the aggregate lending scale D and the intervention scale Φ as given, so do they take ϕ.

By injecting fiat money into the corn market, the intervention increases the bad-state nominal corn price $1/p_b$, and thence entrepreneurs' sale revenue, and thence banks' profit margin γ_b. More strictly, without intervention, $\gamma_b = 0$. As said above, what happen in the Phase is as follows. In the bad state, money is cheap and banks find corn too expensive. They are thus unwilling to issue bank money for their shareholders to buy corn. The only buyers are entrepreneur-employed workers, who supply D units of money, just sufficient for entrepreneurs to replay the loan principals. Now with the intervention, money is even cheaper, corn even more expensive; hence even stronger is banks' unwillingness to exchange money for corn. However, in addition to the D units of bank money that workers supply, the central bank injects $D\phi$ units of fiat money to buy corn. The aggregate supply of money on the corn market is thus $D(1+\phi)$. Given that the aggregate entrepreneurs' demand of money is no greater than $D(1+r)$, if the central bank over-floods the market with fiat money so $\phi > r$, the real value of money $p_b = 0$, which, we will show, is never an optimal scale. Therefore, the intervention scale $\phi \leq r$. It follows that all these $D(1+\phi)$ units of money first flow to entrepreneurs as sales revenue and then to banks as loan repayments. The intervention thus raises the bad-state nominal profit margin of bank lending from 0 to

$$\gamma_b = \phi. \tag{39}$$

Now consider the real effects of the intervention. With the bad-state profit margin γ_b rise, the good-state margin $\gamma_g = r$ should fall, which, we will show, is indeed the case for most of the times. The fall in the bank lending rate has a positive effect for entrepreneus' project scale. However, this is not the only real effect of the intervention. Because it floods the corn market with fiat money, it reduces the real value p_b of money and thereby $\tau = p_b/p_g$. This raises the borrowing cost and has a negative effect for the project scale: As was explained in the discussion of Equation (9), in the default regime, the ratio $p_g/p = [q + (1-q)\,\tau]^{-1}$ compounds entrepreneurs' borrowing cost and a lower τ pushes up this compounding factor. The two channels therefore generate two offsetting effects. What is the direction of the net effect? The answer depends on the winding-down mechanism, we show below in a strict analysis of the policies. Observe that, *for both policies*, the effect of the intervention for the corn market clearing has been represented by Equation (39). Only its effect for the decision of individual banks is left to be examined.

5.1 Policy 1

Consider the representative bank. In the good state, there is no policy intervention and the analysis is unchanged. In particular, the nominal dividend is $v_g = h + \gamma_g(r)d - h'_g$. In the bad state, the bank has a fiat-money position $h + \gamma_b(r)d$ before it decides on the next-period position h'_b. With Policy 1, the bank needs to swap $d\phi$ units of fiat money for corn with the central bank at the end of the period. In order to leave a position of h'_b for the next period, the bank needs prepare $h'_b + d\phi$ units of fiat money before the swap. Due to the no-call constraint, these quantity of fiat money must all come from the bank's position. The no-call constraint thus commands:

$$(h + \gamma_b(r)d) - (h'_b + d\phi) \geq 0. \tag{40}$$

The difference on the left hand side of Inequality (40) is paid out as dividend. In addition, the bank obtains $d\phi p_b$ units corn from the swap with the central bank. They are distributed to the shareholders, given that corn is perishable. Thus the real dividend in the bad state is $[(h + \gamma_b(r)d) - (h'_b + d\phi)]\,p_b + d\phi p_b = [h + \gamma_b(r)d - h'_b]\,p_b$. The bank's problem is thus:

$$V(h) = \max_{d,\{h'_s\}_{s=g,b}} \mathbf{E}\left(\left[h + \gamma_s(r)d - h'_s\right] p_s + \beta V(h'_s)\right), \tag{41}$$

$$\text{s.t. } (40) \text{ and } (1).$$

The effect of Policy 1 on individual banks' decision is given by the following lemma.

Lemma 4 *With Policy 1, the results (i) and (iii) of Lemma 2 still hold, but result (ii) – Equation (17) – is changed to*

$$\mathbf{E}_s\left(\gamma_s\left(r\right)\right) = r_f + \left(1-q\right)\phi\left(1-\tau\right). \tag{42}$$

Proof. See Appendix. ■

The extra term on the right hand side of (42), $(1-q)\,\phi\,(1-\tau)$, is due to the fact that the winding-down mechanism of Policy 1 engenders a *marginal* cost to bank lending: To lending out one more unit of bank money, the bank is obliged to acquire ϕ more unit of fiat money in the bad state, which tightens the no-call constraint (40), incurring an expected real cost of $(1-q)\,\mu p_b \times \phi|_{(18)} = (1-q)\,\phi\,(1-\tau)\,p_g$. In net of this extra marginal cost, the bad-state net profit margin, denoted by γ_b^{net}, becomes

$$\gamma_b^{net} = \gamma_b - \phi\left(1-\tau\right)|_{(39)} = \phi\tau. \tag{43}$$

Still the good-state net profit margin $\gamma_g^{net} = r$. Then, Equation (42) is equivalent to

$$qr + \left(1-q\right)\phi\tau = r_f, \tag{44}$$

or alternatively, $\mathbf{E}_s\left(\gamma_s^{net}\right) = r_f$. That is, the expected the net marginal profit margin is equal to the required risk-free lending rate, parallel to Equation (17). The difference is that now r is negatively related to the bad-state *net* profit margin γ_b^{net} rather than the margin γ_b:

$$r = \frac{r_f - \left(1-q\right)\gamma_b^{net}}{q}. \tag{45}$$

Let r_{P1} and τ_{P1} denote, respectively, the lending rate and the inverse price fluctuation with Policy 1. Intuitively, if in the bad state the central bank flood the corn market with more fiat money, the real value p_b should fall further and so should τ_{P1}; that is,

$\tau'_{P1}(\phi) < 0$. However, that clouds the sign of $d\gamma_b^{net}/d\phi$ and hence that of $r'_{P1}(\phi)$ (the two terms, by (45), have opposite signs). The sign of $r'_{P1}(\phi)$ is indeed not constant, as we show below.

Nowe we determine (r_{P1}, τ_{P1}). First, consider the case where the intervention scale ϕ is not too big. Without interventions, $\tau^* > \theta(\sigma)$. If the intervention scale ϕ is small enough, it drag τ down only a little bit, so that $\tau_{P1} > \theta(\sigma)$ still. In this case, $\gamma_b = (\sigma\alpha\tau)^{-1}(1+r) - 1$ by (12), which together with Equation (39), leads to

$$(\sigma\alpha\tau)^{-1}(1+r) - 1 = \phi. \tag{46}$$

Then, Equations (46) and (44) lead to:

$$r_{P1} = \frac{r^* - \frac{(1-q)\phi}{q(1+\phi)\sigma\alpha}}{1 + \frac{(1-q)\phi}{q(1+\phi)\sigma\alpha}} \tag{47}$$

$$\tau_{P1} = \frac{\tau^*}{1 + \left(1 + \frac{(1-q)}{q\sigma\alpha}\right)\phi}. \tag{48}$$

It is straightforward to see that if $\phi \to 0$, (r_{P1}, τ_{P1}) goes back to (r^*, τ^*), the values with no interventions. Moreover, $\tau'_{P1}(\phi) < 0$ as was intuitively argued. And $r'_{P1}(\phi) < 0$ as well, because in the present case $\gamma_b^{net} = \phi\tau_{P1}(\phi)$ increaseswith ϕ. With (48), $\tau_{P1} > \theta(\sigma)$ if and only if $\phi < \phi_d$, where the threhold

$$\phi_d := \frac{r_f - q(\theta(\sigma)\sigma\alpha - 1)}{\theta(\sigma)[q\sigma\alpha + (1-q)]}. \tag{49}$$

If the intervention scale is small enough – $\phi < \phi_d$ – entrepreneurs remain in the default regime, we have just shown. Now consider the other extreme. Is it possible that the intervention floods the market with so much fiat money, and thus increase the nominal revenue of entrepreneurs so much, that none of them defaults in the bad state? If that is the case, $\gamma_b = r$ by (12), which together with Equations (39) implies $r = \phi$. This substituted into (42) leads to:

$$r_{P1} = \phi \tag{50}$$

$$\tau_{P1} = \frac{r_f - q\phi}{(1-q)\phi}. \tag{51}$$

Then, indeed $\tau_{P1} < \theta(\sigma)$ – and hence no entrepreneurs default – if and only if $\phi > \phi_{nd}$, where

$$\phi_{nd} := \frac{r_f}{q + (1-q)\theta(\sigma)}. \tag{52}$$

Moreover, $\tau_{P1} > 0$ so that $p_b > 0$ if and only if $\phi < \overline{\phi}_{P1}$, where

$$\overline{\phi}_{P1} = \frac{r_f}{q}.$$

That is, if ϕ goes above ϕ_{nd}, the upper bound it can go $\overline{\phi}_{P1}$; with a scale above the bound, the intervention would flood the corn market with so much fiat money as to make it worthless.

Therefore, if the intervention is of a scale large enough – $\phi > \phi_{nd}$ (but still below $\overline{\phi}_{P1}$)– then it eliminates default on bank loans no matter how low the productivity A_b goes. However, that might not be a good idea in terms of real efficiency. Observe that $r'_{P1}(\phi) > 0$; that is because in this scenario, τ_{P1} decreases with ϕ so fast that the net bad-state profit margin $\tau_{P1}\phi$ actually decreases with ϕ. As a result, the efficiency decreases with ϕ in this scenario, so it is never improves efficiency to let ϕ go beyond ϕ_{nd}.

We have found two thresholds of the policy scale ϕ. They are compared in the lemma below.

Lemma 5 *$\phi_d > 0$ if $\sigma > \sigma_{c1}$ and $\phi_{nd} > \phi_d$.*

Proof. See Appendix. ■

We have seen that $\tau_{P1} > \theta(\sigma)$, and so all entrepreneurs default in the bad state, if and only if $\phi < \phi_d$; and $\tau_{P1} < \theta(\sigma)$, and so no entrepreneurs default in the bad state, if and only if $\phi > \phi_{nd}$. In between, if $\phi \in [\phi_d, \phi_{nd}]$, then it must be a case where

$$\tau_{P1} = \theta(\sigma) \tag{53}$$

and entrepreneurs are indifferent between the default and no-default scales in the bad state and play a mixed strategy. Suppose fraction ξ of them chooses the default scale l^d, the rest $1 - \xi$ fraction the no-default scale l^{nd}. Banks' profit margin of lending to the former is $(1 + r) / (\sigma\alpha\tau) - 1$ by (10) and to the latter r. Then, the average profit margin of bank lending in the bad state is

$$\gamma_b = \xi \left(\frac{1 + r}{\sigma\alpha\theta(\sigma)} - 1 \right) + (1 - \xi) r, \tag{54}$$

which, together with $\gamma_b = \phi$ (from 39), leads to $\xi = \xi(r, \phi)$, where

$$\xi(r,\phi) := \frac{1}{1-(\sigma\alpha\theta(\sigma))^{-1}}\left(1-\frac{1+\phi}{1+r}\right). \tag{55}$$

The lending rate, with (53) substituted into (44), is found as follows:

$$r_{P1} = \frac{r_f - (1-q)\,\theta(\sigma)\,\phi}{q}. \tag{56}$$

Then, the default fraction $\xi = \xi(r_{P1}(\phi), \phi)$. It decreases with ϕ;[11] intuitively, a larger scale intervention injects more money into the corn market and increases entrepreneurs' sales revenue, so fewer of them are insolvent. Moreover, $\xi = 1$ at $\phi = \phi_d$ and $\xi = 0$ at $\phi = \phi_{nd}$. Therefore the effect of Policy 1 is continuous.

Now we determine how the normalized project scale with Policy 1, denoted by e_{P1}, depends on the intervention scale ϕ. If $\phi < \phi_d$ or $\phi > \phi_{nd}$, we have $\tau_{P1} > \theta(\sigma)$ or $\tau_{P1} < \theta(\sigma)$ and then $e_{P1} = e(\tau_{P1}, r_{P1})$, where function $e(\tau, r)$ is given by Equation (9). If $\phi \in [\phi_d, \phi_{nd}]$, fraction $\xi = \xi(r_{P1}(\phi), \phi)$ of entrepreneurs opts for l^d, $1-\xi$ for l^{nd}. Hence the average project scale $l = \xi l^d + (1-\xi)\, l^{nd}$. With l^d and l^{nd} given in Lemma 1, in this mixed-strategy case, $e_{P1} = \widetilde{e}(r_{P1}(\phi), \phi)$, where

$$\widetilde{e}(r,\phi) := \frac{1}{1+r}\left(\xi(r,\phi)\left[\left(\frac{q+(1-q)\,\theta(\sigma)}{q+(1-q)\,\sigma^{-1}}\right)^{\frac{1}{1-\alpha}} - 1\right]+1\right)^{1-\alpha}. \tag{57}$$

Altogether,

$$e_{P1}(\phi) = \begin{cases} \frac{1}{\sigma\alpha\left(r_f+q\right)\left(q+(1-q)\sigma^{-1}\right)}\left(q\left(q\sigma\alpha+1-q\right)+\frac{(1-q)r_f}{1+\phi}\right), & \text{if } \phi \in [0, \phi_d]; \\ \widetilde{e}(r_{P1}(\phi), \phi), & \text{if } \phi \in [\phi_d, \phi_{nd}]; \\ \frac{1}{1+\phi}, & \text{if } \phi \in \left[\phi_{nd}, \overline{\phi}_{P1}\right]. \end{cases}$$

Now consider the optimal scale ϕ_1^* of the policy. With Assumption 2 given in (37), entrepreneurs' projects are always below the first best scale. The optimal intervention scale ϕ_1^* is therefore the one that maximises the normalised project scale e_{P1}; that is,

$$\phi_1^* = \arg\max_{\phi\in\left[0,\overline{\phi}_{P1}\right]} e_{P1}(\phi). \tag{58}$$

We can see that $e'_{P1} < 0$ for $\phi \in [0, \phi_d]$ and $\phi \in \left[\phi_{nd}, \overline{\phi}_{P1}\right]$. The mixed-strategy case is not so straightforward. That is because in this case, a larger scale of the policy

[11] $[\xi(r_{P1}(\phi), \phi)]'_\phi < 0$ because $\xi'_r > 0$, $r'_{P1} < 0$ (see 56) and $\xi'_\phi < 0$.

intervention produces two conflicting effects. On the one hand, a bigger ϕ reduces the lending rate $r_{P1}(\phi)$ (by 56), which has a positive effect for the project scale. On the other hand, we have seen a bigger ϕ induces fewer entrepreneurs to choose the default scale l^d, more the no-default scale l^{nd}, which reduces the average project scale because $l^d > l^{nd}$ by Lemma 1. To determine the net effect, we make the following assumption.

Asssumption 3: $\alpha \geq \frac{1}{2}$ and

$$1 - \frac{1}{2}\alpha(1-q) - \frac{1}{2q - (1-\alpha)(1-q)} \geq 0. \tag{59}$$

Because the right hand side of the weak inequality (59) increases with q, the inequality is equivalent to $q \geq \underline{q}$ for some $\underline{q} \in (1/2, 1)$.

Lemma 6 *Under Assumption 3, if $\sigma \geq \sigma_{c1}$, then*

$$\frac{\partial \widetilde{e}(r,\phi)}{\partial r} > 0. \tag{60}$$

Proof. See Appendix. ■

Observe that $\widetilde{e}'_\phi < 0$ because $\partial \xi(r,\phi)/\partial \phi < 0$ by (55). Then, for $\phi \in [\phi_d, \phi_{nd}]$, we also have

$$e'_{P1}(\phi) = \underset{+}{\widetilde{e}'_r} \cdot \underset{-}{r'_{P1}}(\phi) + \underset{-}{\widetilde{e}'_\phi} < 0.$$

Therefore, $e'_{P1}(\phi) < 0$ for all $\phi \in \left[0, \overline{\phi}_{P1}\right]$. Hence, the optimal scale of Policy 1 – the solution to Problem (58) – is $\phi_1^* = 0$. To summarize.

Proposition 2 *The nominal intervention Policy 1 produces real effects, but the optimal scale of the policy is 0.*

The failure of Policy 1 to make any improvement is due to its winding-down mechanism. To see this point, consider the case where the intervention scale $\phi < \phi_d$. In this case, Policy 1 reduces both r and τ. The reduction in r has a positive effect for efficiency, that in τ a negative one, but the former is dominated by the latter. The reason is that the positive effect is attenuated by the winding-down mechanism, whereby for *each* unit of bank money lend out, banks are demanded to exchange ϕ unit of fiat money for corn at the rate of 1 to p_b. This exchange is costly to banks, for which at this exchange rate, fiat money is undervalued, corn overvalued; indeed, on their own, they abstain from the exchange. Therefore, the winding-down mechanism of Policy 1 engenders a marginal cost of lending, and thereby reduces the scale in which Policy 1 increases the net bad-state profit margin, from ϕ to $\phi\tau$ (Equation 43). It is by this increase that the policy reduces the lending rate r (see Equation 45). Consequently, the pace in which r is reduced is slowed, the positive effect attenuated.

Policy 2 is wound down in a different mechanism, whereby the quantity of fiat money that banks are demanded to exchange for corn is fixed, regardless of their lending scales. As a result, while this exchange is still costly to banks, the cost is now a *fixed cost* instead of a marginal cost. Consequently the positive effect is not attenuated and Polcy 2 has a chance to improve efficiency, as we see below.

5.2 Policy 2

Again, in the good state, nothing changes and the real dividend is $\left(h + \gamma_g d - h'_g\right) p_g$. In the bad sate, now banks are demanded to exchange a fixed quantity φ of fiat money for corn at the winding-down stage of the policy. In order to leave a position h'_b for the next period, the representative bank need have $h'_b + \varphi$ units of fiat money before the exchange. All this fiat money come from the bank's own book by the no-call constraint, which, in parallel to (40), is now:

$$(h + \gamma_b d) - (h'_b + \varphi) \geq 0. \tag{61}$$

Similarly, the bad-state real dividend is $c_b = \left[(h + \gamma_b d) - (h'_b + \varphi)\right] p_b + \varphi p_b = (h + \gamma_b d - h'_b)\, p_b$. The bank's problem is thus:

$$V(h) = \max_{d, \{h'_s\}_{s=g,b}} \mathbf{E}\left(\left(h + \gamma_s(r) d - h'_s\right) p_s + \beta V(h'_s)\right),$$
$$\text{s.t. } (61) \text{ and } (1).$$

The effect of this policy on individual banks' decision is given by the following lemma.

Lemma 7 *With Policy 2, all the three results of Lemma 2 still hold. In particular, Equation (17) holds:*

$$\mathbf{E}_s\left(\gamma_s\left(r\right)\right) = r_f. \tag{62}$$

Proof. See Appendix. ■

In Equation (62), unlike in Equation (42), its counterpart with Policy 1, the term $(1-q)\,\phi\,(1-\tau)$ is absent. The term represents the marginal cost that the winding-down mechanism of Policy 1 engenders. It is absent in (62), because the winding-down mechanism of Policy 2 engenders a fixed cost instead of a marginal cost, as we have intuitively argued.

With $\gamma_b = \phi$ by Equation (39) and $\gamma_g = r$, it follows from (62) that the lending rate with Policy 2 is

$$r_{P2} = \frac{r_f - (1-q)\,\phi}{q}. \tag{63}$$

Moreover, by this equation, we find the upper bound $\overline{\phi}_{P2}$ of the scale of Policy 2. Because $r_{P2} = \gamma_g \geq r_f$, (63) implies $\phi \leq r_f$. Hence,

$$\overline{\phi}_{P2} = r_f.$$

Obviously, $r'_{P2}\left(\phi\right) < 0$. Indeed, as we have intuitively argued, under Policy 2, the lending rate decreases in a faster pace than it does under Policy 1.

Lemma 8 $r'_{P2}\left(\phi\right) < r'_{P1}\left(\phi\right)$.

Proof. See Appendix. ■

We are left to find τ_{P2}, the inverse price fluctuation with Policy 2. In parallel to the analysis of Policy 1, first consider the case where ϕ is small enough so that $\tau_{P2} > \theta\left(\sigma\right)$ holds. In this case, $\gamma_b = \left(\sigma\alpha\tau\right)^{-1}\left(1 + r_{P2}\right) - 1$ by (12). Then, that $\gamma_b = \phi$, together with Equation (63), leads to:

$$\tau_{P2} = \frac{1}{1+\phi}\left[\tau^* - \frac{1-q}{q\sigma\alpha}\phi\right]. \tag{64}$$

It follows that $\tau_{P2} > \theta\left(\sigma\right)$ if and only if $\phi < \widetilde{\phi}_d$, where $\widetilde{\phi}_d$ is the root of

$$\tau_{P2}\left(\widetilde{\phi}_d\right) = \theta\left(\sigma\right). \tag{65}$$

Lemma 9 $\widetilde{\phi}_d \in (0, r_f)$ *if* $\sigma \geq \sigma_{c2}$.

Proof. See Appendix. ■

Second, consider the possiblity that if ϕ is big enough, the intervention lifts entrepreneurs into the no-default regime. If that is the case, $\gamma_b = r_{P2}$. Then, that $\gamma_b = \phi$ together with Equation (63) leads to $\phi = r_f = \overline{\phi}_{P2}$. Hence, unlike Policy 1, Policy 2 can save entrepreneurs from default in the bad state only at one particular intervention scale, namely, the maximum scale.

In between, if $\phi \in \left[\widetilde{\phi}_d, \overline{\phi}_{P2}\right]$, as with Policy 1, is the mixed-strategy case where $\tau_{P2} = \theta(\sigma)$ and a fraction ξ of entrepreneurs opts for l^d, $1-\xi$ for l^{nd}. Following the same analysis with Policy 1, $\xi = \xi(r_{P2}(\phi), \phi)$, where $\xi(r, \phi)$ is given by (55). With $r_{P2}(\phi)$ given by (63),

$$\xi = \frac{r_f - \phi}{(r_f + q - (1-q)\phi)\left(1 - (\sigma\alpha\theta(\sigma))^{-1}\right)}. \tag{66}$$

Again, as with Policy 1, ξ decreases with ϕ from 1 to 0 over $\left[\widetilde{\phi}_d, \overline{\phi}_{P2}\right]$, so if ϕ is at the upper end $\overline{\phi}_{P2}$, indeed entrepreneurs all choose l^d.

Now we determine how the normalized project scale with Policy 2, denoted by e_{P2}, depends on the policy scale ϕ. If $\phi < \widetilde{\phi}_d$, then $\tau_{P2} > \theta(\sigma)$ and $e_{P2} = e(\tau_{P2}, r_{P2})$, where $e(\tau, r)$ is given by (9). Then,

$$e_{P2} = \frac{1}{1+r_f}\eta(\tau_{P2}(\phi), \sigma), \tag{67}$$

where $\eta(\tau, \sigma)$ is given by (35). For the mixed-strategy case, where $\phi \in \left[\widetilde{\phi}_d, \overline{\phi}_{P2}\right]$, following the same analysis with Policy 1, we find $e_{P2} = \widetilde{e}(r_{P2}(\phi), \phi)$, where $\widetilde{e}(r, \phi)$ is given by (57). Together,

$$e_{P2}(\phi) = \begin{cases} \frac{1}{1+r_f}\eta(\tau_{P2}(\phi), \sigma), & \text{if } \phi \in \left[0, \widetilde{\phi}_d\right]; \\ \widetilde{e}(r_{P2}(\phi), \phi), & \text{if } \phi \in \left[\widetilde{\phi}_d, \overline{\phi}_{P2}\right]. \end{cases}$$

The optimal scale ϕ_2^* of Policy 2 solves the following problem:

$$\phi_2^* = \arg \max_{\phi \in \left[0, \overline{\phi}_{P2}\right]} e_{P2}(\phi). \tag{68}$$

Again, we apply Lemma 6 for the mixed-strategy case. For $\phi \in \left[\widetilde{\phi}_d, \overline{\phi}_{P2}\right]$, we have

$$e'_{P2}(\phi) = \underset{+}{\widetilde{e}'_r} \cdot \underset{-}{r'_{P2}(\phi)} + \underset{-}{\widetilde{e}'_\phi} < 0. \tag{69}$$

This result has two implications. First, $e_{P2}\left(\widetilde{\phi}_d\right) > e_{P2}\left(\overline{\phi}_{P2}\right)$. We find $e_{P2}\left(\overline{\phi}_{P2}\right) = e_{P2}(r_f) = 1/(1+r_f)$ because at $\phi = \overline{\phi}_{P2}$, the no-default regime prevails so that $e = 1/(1+r_{P2})$ and $r_{P2} = r_f$ by (63). It follows that

$$e_{P2}\left(\widetilde{\phi}_d\right) > \frac{1}{1+r_f}. \tag{70}$$

Recall from Figure 4 that without interventions, the efficiency level $e^* = 1/(1+r_f)$ in Phase 1 and falls below it in Phase 3 if $\sigma > \sigma_e$. By Inequality (70), the intervention of Policy 2 gurantees that the effiency level anywhere in Phase 3 is higher than that in Phase 1 where entrepreneurs never default so there is no scope for interventions.

The second implication of (69) is that the optimal policy scale is within $\left[0, \widetilde{\phi}_d\right]$; that is,

$$\phi_2^* = \arg \max_{\phi \in \left[0, \widetilde{\phi}_d\right]} \frac{1}{1+r_f} \eta\left(\tau_{P2}(\phi), \sigma\right). \tag{71}$$

Proposition 3 *The nominal operation Policy 2 produces real effects. The optimal scale is*

$$\phi_2^* = \begin{cases} 0, & \text{if } \sigma \in [\sigma_{c2}, \sigma_p]; \\ \widetilde{\phi}_d, & \text{if } \phi \in [\sigma_p, \infty), \end{cases} \tag{72}$$

where σ_p is the unique root of

$$\theta(\sigma_p) = \frac{q}{r_f + q}. \tag{73}$$

Moreover $\sigma_p \in (\sigma_{c2}, \sigma_e)$, where σ_e is given by Proposition 1.

Proof. See Appendix. ■

By the proposition, Policy 2 can increase the scale of entrepreneurs' projects if and only if the productivity fluctuation $\sigma > \sigma_p$. Intuition is as follows. Recall that the intervention generates two offsetting effects for the project scale: The positive one by reducing the lending rate r_{P2}, the negative one by reducing the inverse price fluctuation τ_{P2}. The strength of either effect can be measured by the pace of the reduction, that is, the absolute value of $r'_{P2}(\phi)$ and $\tau'_{P2}(\phi)$. Because $|r'_{P2}(\phi)| = (1-q)/q$ by (63), the strength of the positive effect is a constant, whereas that of the negative one decreases with σ, given $|\tau'_{P2}(\phi)| = \frac{1+r_f}{q\alpha(1+\phi)^2}\sigma^{-1}$ by (64). Therefore, the positive effect dominates the negative one if and only if σ is large enough.

If $\sigma > \sigma_p$, the central bank should implement Policy 2 to the point where it starts inducing entrepreneurs to select the no-default scale; going further would reduce the avearge project scale because the no-default scale l^{nd} is smaller than the default scale l^d according to Lemma 1. For $\sigma > \sigma_p$, hence, Policy 2 at the optimal scale raises the efficiency level from e^* to $e^*_{P2}(\sigma) := e_{P2}\left(\widetilde{\phi}_d\right) = \eta\left(\theta\left(\sigma\right), \sigma\right) / \left(1 + r_f\right).$

Lemma 10 *For $\sigma > \sigma_{c1}$, $\frac{de^*_{P2}(\sigma)}{d\sigma} < 0$ and $\lim_{\sigma\to\infty} e^*_{P2}(\sigma) = \frac{1}{1+r_f}$.*

Proof. See Appendix. ■

The efficiency level $e^*(\sigma)$ without interventions is illustrated in Figure 4. Based on this lemma and Proposition 3, the difference made by Policy 2 at the optimal scale, namely, $e^*_{P2}(\sigma)$ versus $e^*(\sigma)$, is illustrated the following figure.

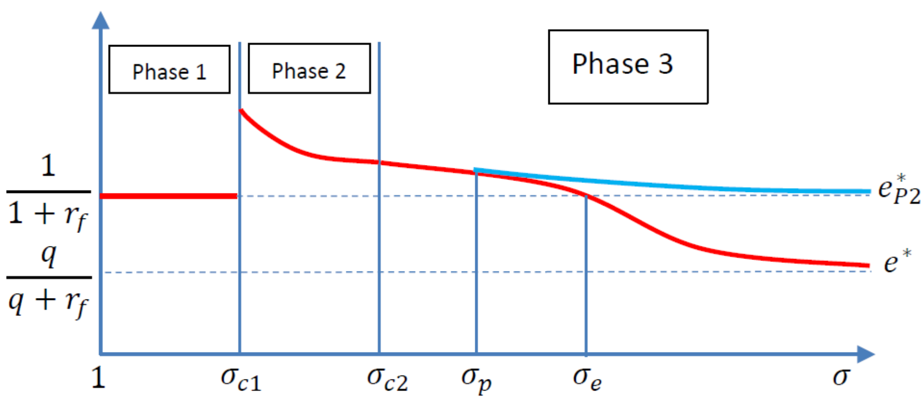

Figure 6: For $\sigma \geq \sigma_p$, Policy 2 at the optimal scale raises the efficiency level from e^* to e^*_{P2}.

6 Conclusion

Typically central banks intervene with asset markets and this type of interventions is what the existing studies of monetary economics have focused on. This paper demonstrates a new possibility: A nominal intervention on a product market produces real effects and, properly designed, improves efficiency. The paper builds on three facts: (1) The major form of means of payment for real goods and services is bank money, not fiat money; (2) The money that a bank lends out is its liability to pay fiat money, which the bank can freely create; and (3) fiat money is necessary for banks to create and lend out bank money. We show that if the productivity fluctuation is large enough, the nominal price fluctuates and the real-sector borrowers – entrepreneurs – default badly when the negative productivity shock hits, unable to make any interest payment on their bank loans. In this scenario, the following intervention on the product market produces real effects: Whenever the negative productivity shock hits, the central bank first prints fiat money to buy the product – corn in the model economy – and then, after the market clears, retires the injected money by demanding banks to exchange a quota of fiat money for corn with the central bank at the same market price.

The mechanism builds on a new effect that this paper discovers, namely that the nominal price fluctuation compounds the funding cost of entrepreneurs on the top of the lending rate if they default upon the negative productivity shock. Because entrepreneurs default in this bad state, they care about the cost of loan repayments only in the good state, upon the positive productivity shock. The real value of money in the good state is higher than the ex ante value when they borrow money. As a result, with each unit of money they borrow, they acquire something worth the ex-ante value, but repay it with the good-state value. The ratio of the latter over the former thus compounds their funding cost. The greater the fluctuation of money's value, the larger the compounding factor and the higher the funding cost of entrepreneurs.

The intervention on the corn market produces real effects by increasing both the price fluctuation and the bank lending profit margin in the bad state. First, by injecting fiat money into the corn market, the intervention raises the nominal price of corn in the bad state and makes it even higher relative to the good-state price, hence *increasing the nominal price fluctuation*. This pushes up the funding cost of entrepreneurs, as said above. Second, as the bad-state nominal product price is raised, so are the nominal sales revenue of entrepreneurs and their repayment to bank loans. Hence, the intervention *increases the nominal profit margin of bank lending* in the bad state. The lending rate, which is negatively related to this profit margin, falls as a result. This pushes down the funding cost of entrepreneurs.

The two channels thus produce conflicting effects. The net effect depends on how the intervention is wound down. It is wound down by demanding banks to exchange a quota of fiat money for corn with the central bank. If the quota for a bank is proportional to its lending scale, then in net the intervention only raises entrepreneurs' funding cost and reduces efficiency. However, if that quota is fixed, independent of the lending scale, then the policy intervention raises efficiency up to an extent if the productivity fluctuation is sufficiently large.

Any nominal intervention, on any market, inevitably causes changes in the nominal prices on the market and also in the nominal profit margin of certain market players. We demonstrate that these changes, though receiving little attention thus far, can have real consequences.

Lastly, we show that banks' money creation reduces the lending rate by leveraging up banks' return to holding fiat money.

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Appendix

Proof of Lemma 1:

Proof. Substitute $M = wl/p$, and the entrepreneur's problem is equivalent to find

$$L^* := \arg\max_l \mathbf{E}_s(\max(A_s l^\alpha - wp_s \frac{1+r}{p} \times l, 0); \tag{74}$$

that is, L^* is the set of the solutions. She might defaults in the bad state, as was argued, which happens if $A_b l^\alpha < wp_b \frac{1+r}{p} \times l \Leftrightarrow$

$$l > l_c := \left(\frac{A_b}{w(1+r)} \times \frac{p}{p_b}\right)^{\frac{1}{1-\alpha}}.$$

The objective function of Problem (74) is therefore

$$V(l) == \begin{cases} V_{nd}(l), & \text{if } l \leq l_c; \\ V_d(l), & \text{if } l \geq l_c, \end{cases}$$

where

$$\begin{aligned} V_{nd}(l) &: = A_e l^\alpha - w(1+r)l; \\ V_d(l) &: = q\left[A_g l^\alpha - wp_g \frac{1+r}{p} l\right]. \end{aligned}$$

We can also find

$$\begin{aligned} l^{nd} &: = \arg\max_l V_{nd}(l) = \left(\frac{A_e \alpha}{w(1+r)}\right)^{\frac{1}{1-\alpha}} \\ l^d &: = \arg\max_l V_d(l) = \left(\frac{A_g \alpha}{w(1+r) \times p_g/p}\right)^{\frac{1}{1-\alpha}}. \end{aligned}$$

Because both $V^{nd}(l)$ and $V^d(l)$ are strictly concave, we have

$$\begin{aligned} V'_{nd} > 0 \text{ if } l < l^{nd}; &\quad V'_{nd} < 0 \text{ if } l > l^{nd} \\ V'_d > 0 \text{ if } l < l^d; &\quad V'_d < 0 \text{ if } l > l^d. \end{aligned} \tag{75}$$

Moreover,

$$\begin{aligned} l^{nd} &\leq l_c \Leftrightarrow \frac{p_b}{p} \leq \frac{A_b}{A_e \alpha} \\ l^d &\leq l_c \Leftrightarrow \frac{p_b}{p} \leq \frac{A_b}{qA_g\alpha + (1-q)A_b}. \end{aligned}$$

Observe that $\frac{A_b}{qA_g\alpha+(1-q)A_b} < \frac{A_b}{A_e\alpha}$. Then we have the following three cases.

Case 1: $\frac{p_b}{p} \leq \frac{A_b}{qA_g\alpha+(1-q)A_b} < \frac{A_b}{A_e\alpha}$. In this case, $\max\left(l^d, l^{nd}\right) \leq l_c$. If $l < l^{nd}$, then $l < l_c$; hence $V = V_{nd}$ and $V' = V'_{nd} > 0$ by (75). If $l > l^{nd}$, then either $l < l_c$ still, in which case $V' = V'_{nd} < 0$; or $l > l_c$, in which case $l > l^d$ and $V' = V'_d < 0$; or $l = l_c$, in which case V'_- is not equal to V'_+, but both are negative. Therefore, $L^* = \left\{l^{nd}\right\}$.

Case 2: $\frac{A_b}{qA_g\alpha+(1-q)A_b} < \frac{A_b}{A_e\alpha} \leq \frac{p_b}{p}$. In this case $\min\left(l^d, l^{nd}\right) \geq l_c$. If $l > l^d$, then $l > l_c$ and hence $V' = V'_d < 0$. If $l < l^d$, then, apart from the knife-edge case of $l = l_c$, either either $l > l_c$ still, in which case $V' = V'_d > 0$, or $l < l_c$, in which case $l < l^{nd}$ and $V' = V'_{nd} > 0$. Therefore, $L^* = \left\{l^d\right\}$.

Case 3: $\frac{A_b}{qA_g\alpha+(1-q)A_b} < \frac{p_b}{p} < \frac{A_b}{A_e\alpha}$. In this case $l^{nd} < l_c < l^d$. Then both l^{nd} and l^d are a local optimum. $L^* = \left\{l^{nd}\right\}$ if $V\left(l^{nd}\right) > V\left(l^d\right) \Leftrightarrow V_{nd}\left(l^{nd}\right) > V_d\left(l^d\right) \Leftrightarrow$

$$\begin{aligned} A_e\left(\frac{A_e\alpha}{w(1+r)}\right)^{\frac{\alpha}{1-\alpha}} &> qA_g\left(\frac{A_g\alpha}{w(1+r)} \times \frac{p}{p_g}\right)^{\frac{\alpha}{1-\alpha}} \Leftrightarrow \\ (A_e)^{\frac{1}{1-\alpha}} &> q\left(A_g\right)^{\frac{1}{1-\alpha}}\left(\frac{p}{p_g}\right)^{\frac{\alpha}{1-\alpha}} \Leftrightarrow \\ \frac{p_b}{p} &< \frac{1-\left(\frac{qA_g}{A_e}\right)^{\frac{1}{\alpha}}}{1-q}. \end{aligned}$$

Similarly, $L^* = \left\{l^d\right\}$ if $\frac{p_b}{p} > \frac{1-\left(\frac{qA_g}{A_e}\right)^{\frac{1}{\alpha}}}{1-q}$. And if $\frac{p_b}{p} = \frac{1-\left(\frac{qA_g}{A_e}\right)^{\frac{1}{\alpha}}}{1-q}$, both l^{nd} and l^d are a global optimum: $L^* = \left\{l^{nd}, l^d\right\}$.

The threshold $\frac{1-\left(\frac{qA_g}{A_e}\right)^{\frac{1}{\alpha}}}{1-q}$ is within $\left(\frac{A_b}{qA_g\alpha+(1-q)A_b}, \frac{A_b}{A_e\alpha}\right)$. To show that let $x := \frac{1}{\alpha}$ and $y := \frac{A_e}{qA_g}$. Hence, $x > 1$ and $y > 1$. First,

$$\begin{aligned} \frac{A_b}{qA_g\alpha+(1-q)A_b} &< \frac{1-\left(\frac{qA_g}{A_e}\right)^{\frac{1}{\alpha}}}{1-q} \Leftrightarrow \\ y^x &> 1+(y-1)x, \end{aligned}$$

which holds true because if $f(y) := y^x-(y-1)x-1$, then $f(1) = 0$ and $f' = x(y^{x-1}-1) > 0$ if $y > 1$ and $x > 1$. Second

$$\begin{aligned} \frac{1-\left(\frac{qA_g}{A_e}\right)^{\frac{1}{\alpha}}}{1-q} &< \frac{A_b}{A_e\alpha} \Leftrightarrow \\ 1-a^x &< (1-a)x, \end{aligned}$$

where $a := y^{-1} \in (0,1)$. The last inequality holds true for $x > 1$ and $0 < a < 1$ because $f(a) := 1-a^x-(1-a)x$ satisfies $f(1) = 0$ and $f'(a) = x(1-a^{x-1}) > 0$.

Altogether, Therefore, if $\frac{p_b}{p} < \frac{1-\left(\frac{qA_g}{A_e}\right)^{\frac{1}{\alpha}}}{1-q} \Leftrightarrow \tau < \frac{q}{1-q}\left[\left(\frac{q\sigma+1-q}{q\sigma}\right)^{\frac{1}{\alpha}} - 1\right] = \theta(\sigma)$, then $L^* = \left\{l^{nd}\right\}$. Moreover, $l^{nd} < l_c$ (as $\frac{p_b}{p} < \frac{A_b}{A_e\alpha}$), namely, entrepreneurs will not default in the bad state. If $\tau > \theta(\sigma)$, then $L^* = \left\{l^d\right\}$ and $l^d > l_c$. If $\tau = \theta(\sigma)$, then $L^* = \left\{l^{nd}, l^d\right\}$ and $l^{nd} < l_c < l^d$. ■

Proof of Lemma 2:

Proof. Recall that $(1-q)\mu$ be the multiplier for Constraint $c_b \geq 0$; let λp be that for the scale constraint $\chi h \geq d$. With c_s substituted using the budget constraint (14), the Lagrangean of problem (13) is:

$$\begin{aligned}\mathcal{L} &= \mathbf{E}_s\left((h + d\gamma_s - h_s')p_s + \beta V(h_s')\right) + \\ &\quad \lambda \times (\chi h - d)\, p + (1-q)\,\mu \times (h + \gamma_b(r)d - h_b')p_b.\end{aligned}$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial h_g'} = -p_g + \beta\left[p + \lambda\chi p + (1-q)\,\mu p_b\right] = 0 \tag{76}$$

$$\frac{\partial \mathcal{L}}{\partial h_b'} = -p_b\,(1+\mu) + \beta\left[p + \lambda\chi p + (1-q)\,\mu p_b\right] = 0 \tag{77}$$

$$\frac{\partial \mathcal{L}}{\partial d} = \mathbf{E}_s\left(\gamma_s p_s\right) - \lambda p + (1-q)\,\mu\gamma_b p_b = 0. \tag{78}$$

From equations (76) and (77) that

$$p_g = p_b\,(1+\mu)\,, \tag{79}$$

that is, (18). It follows that

$$\mu = \frac{1-\tau}{\tau}. \tag{80}$$

Therefore, either $\tau = 1$, or $\mu > 0$ and hence the constraint $h + \gamma_b(r)d - h_b' \geq 0$ is binding:

$$h + \gamma_b(r)d - h_b' = 0.$$

Substituting (79) into (76), we find

$$\frac{\lambda p\chi}{p_g} = \frac{1}{\beta} - 1. \tag{81}$$

This equation is intuitive. At the end of each period, the cost of buying a unit of fiat money is always p_g. The net gain is that a unit of fiat money, by enabling the bank to lend extra χ units of credit, relaxes its scale constraint, the real value of which is $\chi p \times \lambda$. Therefore, the next return rate of holding a fiat money is $\chi p\lambda/p_g$, which, in equilibrium, is equal to the cost of holding fiat money due to time preference $1/\beta - 1$.

Substituting (79) into (78), we find

$$p_g \mathbf{E}_s (\gamma_s (r)) = \lambda p. \tag{82}$$

For an intuition for this equation, first note that by the discussion ensuing (18), *the shadow value of fiat money for relaxing the binding no-call constraint taken into account, the value of money is always p_g, regardless of the realised state s.* Then equation (82) describes the trade-off that determines the lending scale d. On the benefit side, lending out one more unit of credit gains the bank in expectation $\mathbf{E}_s (\gamma_s)$ unit of nomnal profit ex post. Because the value of a unit of money is always p_g ex post, the value of this profit is $p_g \mathbf{E}_s (\gamma_s)$. On the cost side, lending out one more unit of credit tightens the scale constraint ex ante, the real value of which is λp. In equilibrium, the two sides are equalised. Equations (82) and (81) lead to (17).

Lastly, equation (81) leads to

$$\lambda = \frac{r_f}{q + (1-q)\tau}. \tag{83}$$

Therefore, $\lambda > 0$. ■

Proof of Proposition 1:

Proof. In Phase 1, $\gamma_s (r) = r > 0$ for $s = g, b$. This substituted into (17), we find $r^* = r_f$. Because $\gamma_b > 0$, $\tau^* = 1$. Then Condition (25) holds, namely $\tau^* < \theta(\sigma) \Leftrightarrow 1 < \theta(\sigma)$, which, because $\theta'(\sigma) < 1$, is equivalent to $\sigma < \sigma_{c1} := \frac{\frac{1}{q}-1}{\left(\frac{1}{q}\right)^{\alpha}-1}$, where σ_{c1} is the root of

$$\theta(\sigma_{c1}) = 1.$$

Because $\theta(\sigma) > \alpha^{-1}\sigma^{-1}$ for any σ, we have $\sigma_{c1} > \alpha^{-1}$.

In Phase 2, because $\gamma_b > 0$, $\tau^* = 1$ still. Moreover, in Phase 2, $\tau > \theta(\sigma)$. It follows from (12) that

$$\gamma_b = \frac{1+r}{\sigma\alpha\tau} - 1. \tag{84}$$

Given $\tau^* = 1$,

$$\gamma_b = \frac{1+r}{\sigma\alpha} - 1. \tag{85}$$

Substituting this (and $\gamma_g = r$) into (17), we find

$$qr^* + (1-q)\left((1+r^*)(\sigma\alpha)^{-1} - 1\right) = r_f, \tag{86}$$

which leads to

$$r^* = \frac{r_f + (1-q)\left(1-(\sigma\alpha)^{-1}\right)}{q + (1-q)(\sigma\alpha)^{-1}}. \tag{87}$$

Condition (26) is

$$1 > \theta(\sigma) \tag{88}$$
$$\frac{1+r^*}{\sigma\alpha} - 1 > 0. \tag{89}$$

Condition (88) is equivalent to $\sigma > \sigma_{c1}$, we have seen. And Condition (89) is equivalent to

$$\begin{aligned} &1 + r^* > \sigma\alpha && |_{(87)} \Leftrightarrow \\ &1 + \tfrac{r_f + (1-q)\left(1-(\sigma\alpha)^{-1}\right)}{q+(1-q)(\sigma\alpha)^{-1}} > \sigma\alpha && \Leftrightarrow \\ &\sigma < \tfrac{1}{\alpha}\left(\tfrac{r_f}{q} + 1\right) = \sigma_{c2}, \end{aligned}$$

where

$$\sigma_{c2} := \frac{1}{\alpha}\left(\frac{r_f}{q} + 1\right) = \frac{1}{\alpha}\frac{\frac{1}{\beta} - 1 + \chi q}{\chi q}.$$

Under Assumption 1, $\sigma_{c2} > \sigma_{c1}$. Hence, if $\sigma \in (\sigma_{c1}, \sigma_{c2})$, both conditions (88) and (89) are met, so the steady state is in Phase 2.

In Phase 3, $\gamma_b = 0$, so (17) becomes

$$qr^* = r_f. \tag{90}$$

In the phase, as in Phase 2, $\tau > \theta(\sigma)$ and hence γ_b is given by (84). That $\gamma_b = 0$ is equivalent to:

$$\frac{1+r}{\sigma\alpha\tau^*} - 1 = 0. \tag{91}$$

Equations (90) and (91) lead to

$$r^* = \frac{1}{q} r_f \tag{92}$$
$$\tau^* = \tau_3^*(\sigma) := \frac{r_f + q}{q\alpha\sigma}. \tag{93}$$

Observe that $\tau_3^*(\sigma) \le 1 \Leftrightarrow \sigma \ge \frac{1}{\alpha}\left(\frac{r_f}{q} + 1\right) = \sigma_{c2}$ and

$$\tau_3^*(\sigma_{c2}) = 1. \tag{94}$$

Condition (27) is equivalent to

$$\begin{aligned} \tau_3^*(\sigma) &> \theta(\sigma) & (95) \\ \gamma_b(r^*) &= 0. & (96) \end{aligned}$$

By Part (1) of Equilibrium Definition 1, Condition (95) follows from $\tau_3^*(\sigma) < 1 \Leftrightarrow \sigma < \sigma_{c2}$. Lemma A1 below shows that Condition (96) holds if $\sigma \geq \sigma_{c2}$ under Assumption 1. therefore, if $\sigma > \sigma_{c2}$, both of the conditions hold and the steady state is in Phase 3.

Lemma A1: Under Assumption 1, $\tau_3^*(\sigma) > \theta(\sigma)$ if $\sigma \geq \sigma_{c2}$.

Proof: Let $x := \sigma^{-1}$ and $\widetilde{\tau}(x) = \frac{1}{\alpha}\frac{\frac{1}{\beta}-1+\chi q}{\chi q}x$ and $\widetilde{\theta}(x) = \frac{q}{1-q}\left[\left(1+\frac{1-q}{q}x\right)^{\frac{1}{\alpha}}-1\right]$. It suffices to prove that $\widetilde{\tau}(x) > \widetilde{\theta}(x)$ for $x \in \left(0, \sigma_{c2}^{-1}\right)$. Obviously $\widetilde{\tau}(0) = \widetilde{\theta}(0) = 0$. At $x \gtrsim 0$, $\widetilde{\theta}(x) \approx \frac{1}{\alpha}x < \widetilde{\tau}(x)$, while if $x \to \infty$, $\widetilde{\theta}(x) = O\left(x^{\frac{1}{\alpha}}\right) > \widetilde{\tau}(x)$. Therefore, $\widetilde{\tau}(x) = \widetilde{\theta}(x)$ has a root within $(0,\infty)$. Moreover, because $\widetilde{\tau}(x)$ is linear and $\widetilde{\theta}(x)$ convex, the root is unique. Denote the root by x_r. Then for $x \in (0, x_r)$, $\widetilde{\tau}(x) > \widetilde{\theta}(x)$. To prove the lemma, it suffices to prove that $x_r > \sigma_{c2}^{-1}$, which, given the uniqueness of the root, is equivalent to $\widetilde{\tau}\left(\sigma_{c2}^{-1}\right) > \widetilde{\theta}\left(\sigma_{c2}^{-1}\right) \Leftrightarrow \tau_3^*(\sigma_{c2}) > \theta(\sigma_{c2})|_{(94)} \Leftrightarrow 1 > \theta(\sigma_{c2})|_{\theta(\sigma_{c1})=1} \Leftrightarrow \theta(\sigma_{c1}) > \theta(\sigma_{c2}) \Leftrightarrow \sigma_{c1} < \sigma_{c2}$, which is Assumption 1. q.e.d.

Now find $e^* = e(\tau^*, r^*)$. If $\sigma < \sigma_{c1}$, the steady state is in Phase 1 and $\tau^* < \theta(\sigma)$, so $e^* = \frac{1}{1+r_f}$. If $\sigma > \sigma_{c1}$, $\tau^* > \theta(\sigma)$, so (84) holds. Substitute (84) into (17) and add 1 on both sides, with some rearrangement, and we have

$$\begin{aligned} (1+r)\left[q+(1-q)(\sigma\alpha\tau)^{-1}\right] &= 1+r_f \Leftrightarrow \\ 1+r &= \frac{1+r_f}{q+(1-q)(\sigma\alpha\tau)^{-1}}. \end{aligned}$$

Substitute this for r into $e(\tau, r)$ in (9) for the case of $\tau > \theta(\sigma)$, the efficiency index in the default regime e^d below:

$$e^d = \frac{\eta(\tau,\sigma)}{1+r_f}, \qquad (97)$$

where, as given by (35),

$$\eta(\tau,\sigma) := \frac{\left[q+(1-q)(\sigma\alpha\tau)^{-1}\right]\left[q+(1-q)\tau\right]}{q+(1-q)\sigma^{-1}}.$$

Therefore, If $\sigma > \sigma_{c1}$,

$$e^* = \frac{\eta(\tau^*(\sigma),\sigma)}{1+r_f}.$$

In Phase 2 $e^* = \frac{1}{1+r_f}\frac{q+(1-q)(\sigma\alpha)^{-1}}{q+(1-q)\sigma^{-1}}$ decreases with σ, and so does it in Phase 3, where

$$e^* = \frac{1}{\alpha} \times \frac{q\frac{\alpha q}{q+r_f} + (1-q)\,\sigma^{-1}}{q + (1-q)\,\sigma^{-1}}. \tag{98}$$

It follows from (98) that $\lim_{\sigma\to\infty} e^* = \frac{q}{q+r_f} < \frac{1}{1+r_f}$. To prove that there exists $\sigma_e > \sigma_{c2}$ such that $e^*(\sigma_e) = \frac{1}{1+r_f}$, it suffices to show that $e^*(\sigma_{c2}) > \frac{1}{1+r_f} \Leftrightarrow \eta\left(\tau_3^*(\sigma_{c2}), \sigma_{c2}\right) > 1 \Leftrightarrow |_{(94)} \frac{q+(1-q)\sigma_{c2}^{-1}\alpha^{-1}}{q+(1-q)\sigma_{c2}^{-1}} > 1$, which holds because $\alpha^{-1} > 1$. ■

Proof of Lemma 3:

Proof. We have seen $e^* < 1$ for $\sigma < \sigma_{c1}$ and $e^{*\prime}(\sigma) < 0$ for $\sigma > \sigma_{c1}$. Hence to prove the first claim of the lemma, it suffices to prove that under Assumption 2, $e_2^* < 1$ at $\sigma = \sigma_{c1}$, which is equivalent to

$$\begin{array}{ll} \frac{q+(1-q)(\sigma_{c1}\alpha)^{-1}}{q+(1-q)\sigma_{c1}^{-1}} < 1 + r_f & \Leftrightarrow \\ \frac{(1-q)\sigma_{c1}^{-1}\left(\alpha^{-1}-1\right)}{q+(1-q)\sigma_{c1}^{-1}} < r_f & \Leftrightarrow \\ \chi < \frac{\left(\frac{1}{\beta}-1\right)\left[q+(1-q)\sigma_{c1}^{-1}\right]}{(1-q)\sigma_{c1}^{-1}(\alpha^{-1}-1)}, & \end{array}$$

which, as σ_{c1} is given by (28), is equivalent to Inequality (37).

For the second claim, Inequalities (30) compared to (37), the latter is stronger than the former if

$$\begin{array}{ll} \left(\frac{1}{\beta}-1\right)\frac{\left(\frac{1}{q}\right)^{\alpha}-1}{q\left[\alpha\left(\frac{1}{q}-1\right)-\left(\left(\frac{1}{q}\right)^{\alpha}-1\right)\right]} > \frac{\frac{1}{\beta}-1}{(1-q^{\alpha})\left(\frac{1}{\alpha}-1\right)} & \Leftrightarrow \\ \frac{\left(\frac{1}{q}\right)^{\alpha}-1}{q\left[\alpha\left(\frac{1}{q}-1\right)-\left(\left(\frac{1}{q}\right)^{\alpha}-1\right)\right]} > \frac{1}{(1-q^{\alpha})\left(\frac{1}{\alpha}-1\right)} & \Leftrightarrow \\ (1-q^{\alpha})\left(\frac{1}{\alpha}-1\right) > q\left[\alpha\frac{\frac{1}{q}-1}{\left(\frac{1}{q}\right)^{\alpha}-1} - 1\right] & \Leftrightarrow \\ \frac{1}{\alpha}\left(1+\left(\frac{1}{q}-\left(\frac{1}{q}\right)^{1-\alpha}\right)\left(\frac{1}{\alpha}-1\right)\right) > \frac{\frac{1}{q}-1}{\left(\frac{1}{q}\right)^{\alpha}-1}. & \end{array}$$

Let $x = \frac{1}{q}$, the last inequality is equivalent to $f(x) = \frac{1}{\alpha}\left(1 + (x - x^{1-\alpha})\left(\frac{1}{\alpha}-1\right)\right)(x^{\alpha}-1) - (x-1) > 0$ for $x > 1$. Observe that $f(1) = 0$ and for $x > 1$,

$$\begin{aligned} f'(x) &> \frac{1}{\alpha}\left(1+\left(x-(x)^{1-\alpha}\right)\left(\frac{1}{\alpha}-1\right)\right)\alpha x^{\alpha-1} - 1 \\ &= \frac{1}{\alpha}\left(\alpha x^{\alpha-1} + (x^{\alpha}-1)(1-\alpha)\right) - 1 > 0 \\ &\Leftrightarrow \alpha x^{\alpha-1} + (x^{\alpha}-1)(1-\alpha) > \alpha \\ &\Leftrightarrow \alpha x^{\alpha-1} + (1-\alpha)\,x^{\alpha} > 1, \end{aligned}$$

which holds true for $x > 1$ because $g(x) := \alpha x^{\alpha-1} + (1-\alpha)\,x^{\alpha}$ satisfies $g(1) = 0$ and $g'(x) = \alpha(1-\alpha)\,x^{\alpha-2}(x-1) > 0$ for $x > 1$. ■

Proof of Lemma 4:

Proof. The Lagrangean of problem (13) is:

$$\begin{aligned}\mathcal{L} &= \mathbf{E}_s\left((h+d\gamma_s-h_s')p_s+\beta V(h_s')\right)+\\ &\quad \lambda\times(\chi h-d)\,p+(1-q)\,\mu\times(h+\gamma_b d-h_b'-d\phi)p_b.\end{aligned}$$

The first order conditions are:

$$\frac{\partial\mathcal{L}}{\partial h_g'} = -p_g+\beta\left[p+\lambda\chi p+(1-q)\,\mu p_b\right]=0 \tag{99}$$

$$\frac{\partial\mathcal{L}}{\partial h_b'} = -p_b\,(1+\mu)+\beta\left[p+\lambda\chi p+(1-q)\,\mu p_b\right]=0 \tag{100}$$

$$\frac{\partial\mathcal{L}}{\partial d} = \mathbf{E}_s\left(\gamma_s p_s\right)-\lambda p+(1-q)\,\mu\left(\gamma_b-\phi\right)p_b=0. \tag{101}$$

From equations (99) and (100) that

$$p_g=p_b\,(1+\mu)\,, \tag{102}$$

which leads to

$$\mu=\frac{1-\tau}{\tau}. \tag{103}$$

Therefore, either $\tau=1$, or $\mu>0$ and hence the constraint $h+\gamma_b(r)d-h_b'\geq 0$ is binding, which, with $h=h_b'=H$ in the steady state, leads to $\gamma_b(r)=0$.

Substituting (103) into (99), we find

$$\frac{\lambda p\chi}{p_g}=\frac{1}{\beta}-1. \tag{104}$$

Hence, $\lambda>0$. Substituting (102) into (101), we find

$$p_g\mathbf{E}_s\left(\gamma_s\left(r\right)\right)=\lambda p+(1-q)\,\phi\mu p_b. \tag{105}$$

As in (82), the left hand side of (105) is the marginal benefit for the bank of lending out one more unit of money, the right hand side the marginal cost. But different to (82), with the policy intervention, to the marginal cost one more term contributes: Now to lending out one more unit of money, the bank is obliged to buy ϕ more unit of fiat money in the bad state, which tightens the no-call constraint in the state; hence the term $(1-q)\,\phi\mu p_b$.

Altogether, Equations (103), (82) and (81) lead to (42). ■

Proof of Lemma 5:

Proof. For the first claim, by (49), $\phi_d > 0 \Leftrightarrow$

$$q\left(\theta\left(\sigma\right)\sigma\alpha - 1\right) < r_f. \tag{106}$$

Given $\theta\left(\sigma\right) < 1$ if $\sigma > \sigma_{c1}$, Inequality (106) follows from $q\left(\sigma\alpha - 1\right) \leq r_f$, which is equivalent to $\sigma \leq \frac{1}{\alpha}\left(\frac{r_f}{q} + 1\right) = \sigma_{c2}$ and thus holds if $\sigma \leq \sigma_{c2}$. Inequality (106) also holds for the case of $\sigma > \sigma_{c2}$. In this case, by Lemma A1, $\theta\left(\sigma\right) < \tau_3^*\left(\sigma\right) = \frac{r_f + q}{q\alpha\sigma}$, to which (106) is equivalent.

For the second claim, by (52), $\phi_{nd} = \frac{r_f}{q+(1-q)\theta(\sigma)} = \frac{r_f/q}{1+\frac{1-q}{q}\theta(\sigma)} > \frac{r_f/q}{\theta(\sigma)\sigma\alpha+\frac{1-q}{q}\theta(\sigma)} > \frac{r_f/q-(\theta(\sigma)\sigma\alpha-1)}{\theta(\sigma)\sigma\alpha+\frac{1-q}{q}\theta(\sigma)} = \phi_d$, where both inequalities use the fact that $\theta\left(\sigma\right) > \left(\sigma\alpha\right)^{-1}$ and thus $\theta\left(\sigma\right)\sigma\alpha > 1$. ■

Proof of Lemma 6:

Proof. Let

$$\kappa := \left(\frac{q + (1-q)\,\theta\left(\sigma\right)}{q + (1-q)\,\sigma^{-1}}\right)^{\frac{1}{1-\alpha}}. \tag{107}$$

We have seen $\kappa > 1$ because $\theta\left(\sigma\right) > \frac{1}{\alpha}\sigma^{-1} > \sigma^{-1}$. Then

$$\widetilde{e}\left(r,\phi\right) = \frac{1}{1+r}\left(\xi\left(r,\phi\right)\left(\kappa - 1\right) + 1\right)^{1-\alpha}.$$

We have

$$\begin{aligned}
\frac{\partial}{\partial r}\log\widetilde{e}\left(r,\phi\right) &= (1-\alpha)\frac{\xi_r'\left(\kappa-1\right)}{\xi\left(r,\phi\right)\left(\kappa-1\right)+1} - \frac{1}{1+r} \\
|_{(55)} &= (1-\alpha)\frac{\kappa-1}{\xi\left(r,\phi\right)\left(\kappa-1\right)+1}\frac{1}{1-\left(\sigma\alpha\theta\left(\sigma\right)\right)^{-1}}\frac{1+\phi}{\left(1+r\right)^2} - \frac{1}{1+r}.
\end{aligned}$$

It follows that $\frac{\partial}{\partial r}\widetilde{e}\left(r,\phi\right) > 0$ is equivalent to

$$(1-\alpha)\frac{\kappa-1}{\xi\left(r,\phi\right)\left(\kappa-1\right)+1}\frac{1}{1-\left(\sigma\alpha\theta\left(\sigma\right)\right)^{-1}}\frac{1+\phi}{1+r} > 1. \tag{108}$$

Observe that the right hand side of (108) decreases with r because $\xi_r' > 0$. Therefore, it holds for any r if it holds for the maximum value of r, at which ξ takes the maximum value, which is 1. By (55), if $\xi = 1$, then $\frac{1+\phi}{1+r} = \left(\sigma\alpha\theta\left(\sigma\right)\right)^{-1}$. Hence, inequality (108) holds if

$$\begin{aligned}
(1-\alpha)\frac{\kappa-1}{\kappa}\frac{\left(\sigma\alpha\theta\left(\sigma\right)\right)^{-1}}{1-\left(\sigma\alpha\theta\left(\sigma\right)\right)^{-1}} &> 1 \Leftrightarrow \\
1 - \left(1 - \frac{(1-q)\left(\theta-\sigma^{-1}\right)}{q+(1-q)\,\theta}\right)^{\frac{1}{1-\alpha}} &> \frac{\sigma\alpha\theta - 1}{1-\alpha}.
\end{aligned} \tag{109}$$

Claim 1 below helps us estimate the left hand side of the inequality.

Claim 1: If $y \geq 2$ and $x \in (0,1)$, then $(1-x)^y \leq 1 - yx + \frac{1}{2}y(y-1)x^2$.

Proof: Let $f(x) := \left[1 - yx + \frac{1}{2}y(y-1)x^2\right] - (1-x)^y$. Then $f(0) = 0$ and $f' = -y + y(y-1)x + y(1-x)^{y-1} = y\left[(1-x)^{y-1} + (y-1)x - 1\right]$. The lemma thus follows from $f' \geq 0$, which holds true because $g(x) := (1-x)^{y-1} + (y-1)x - 1$ satisfies $g(0) = 0$ and $g' = -(y-1)(1-x)^{y-2} + (y-1) = (y-1)\left(1-(1-x)^{y-2}\right) \geq 0$ if $y - 2 \geq 0$ and $1 - x \in (0,1)$.

By Assumption 3, $\alpha \geq 1/2$. Hence, $\frac{1}{1-\alpha} \geq 2$. Then, by Claim 1,

$$\left(1 - \frac{(1-q)(\theta - \sigma^{-1})}{q + (1-q)\theta}\right)^{\frac{1}{1-\alpha}} < 1 - \frac{1}{1-\alpha}\frac{(1-q)(\theta-\sigma^{-1})}{q+(1-q)\theta} + \frac{1}{2}\frac{1}{1-\alpha}\frac{\alpha}{1-\alpha}\left(\frac{(1-q)(\theta-\sigma^{-1})}{q+(1-q)\theta}\right)^2.$$

As a result, inequality (109) follows from

$$\begin{aligned}
\frac{1}{1-\alpha}\frac{(1-q)(\theta-\sigma^{-1})}{q+(1-q)\theta} - \frac{1}{2}\frac{1}{1-\alpha}\frac{\alpha}{1-\alpha}\left(\frac{(1-q)(\theta-\sigma^{-1})}{q+(1-q)\theta}\right)^2 &> \frac{\sigma\alpha\theta - 1}{1-\alpha} \Leftrightarrow \\
\frac{(1-q)(\theta-\sigma^{-1})}{q+(1-q)\theta} - \frac{1}{2}\frac{\alpha}{1-\alpha}\left(\frac{(1-q)(\theta-\sigma^{-1})}{q+(1-q)\theta}\right)^2 &> \sigma\alpha\theta - 1. \quad (110)
\end{aligned}$$

Let $x = \sigma^{-1} < \alpha\theta$. By (6)

$$\begin{aligned}
\theta(x) &= \frac{q}{1-q}\left[\left(1 + \frac{1-q}{q}x\right)^{\frac{1}{\alpha}} - 1\right] \Leftrightarrow \\
x(\theta) &= \frac{q}{1-q}\left[\left(1 + \frac{1-q}{q}\theta\right)^{\alpha} - 1\right]. \quad (111)
\end{aligned}$$

Using this variable transformation, we can write Inequality (110) in terms of θ and $x(\theta)$. Inequality (110) is then equivalent to

$$f(x) := \frac{(1-q)(\theta-x)}{q+(1-q)\theta} - \frac{1}{2}\frac{\alpha}{1-\alpha}\left(\frac{(1-q)(\theta-x)}{q+(1-q)\theta}\right)^2 - \left(\frac{\alpha\theta}{x} - 1\right) > 0, \quad (112)$$

for $x = x(\theta)$ given by (111) and hence $x < \alpha\theta$. Observe that

$$\begin{aligned}
f' &= \frac{-(1-q)}{q+(1-q)\theta} + \frac{\alpha}{1-\alpha}\left(\frac{1-q}{q+(1-q)\theta}\right)^2(\theta - x) + \frac{\alpha\theta}{x^2} \\
&> \frac{-(1-q)}{q+(1-q)\theta} + \frac{\alpha\theta}{x^2} \\
|_{x<\alpha\theta} &> \frac{-(1-q)}{q+(1-q)\theta} + \frac{1}{\alpha\theta} \\
&> \frac{-(1-q)}{q+(1-q)\theta} + \frac{1}{\theta} \\
&> 0.
\end{aligned}$$

Therefore, to prove $f(x(\theta)) > 0$, that is (112), it suffices to prove function $f > 0$ at a lower bound of $x(\theta)$. To find a lower bound, the following claim helps.

Claim 2: $(1+x)^{\alpha} > 1 + \alpha x - \frac{\alpha(1-\alpha)}{2}x^2$ for $\alpha \in (0,1)$ and $x > 0$.

Proof: Let $f(x) := (1+x)^{\alpha} - \left(1 + \alpha x - \frac{\alpha(1-\alpha)}{2}x^2\right)$. Then $f(0) = 0$ and $f' = \alpha\left((1+x)^{\alpha-1} - 1 + (1-\alpha)x\right) > 0$, which together imply $f > 0$. To see $g(x) := (1+x)^{\alpha-1} - 1 + (1-\alpha)x > 0$ for $\alpha \in (0,1)$ and $x > 0$, observe that $g(0) = 0$ and $g' = (1-\alpha)\left(1 - (1+x)^{\alpha-2}\right) > 0$.

With Claim 2, it follows from (111) that

$$\begin{aligned}
x(\theta) &> \frac{q}{1-q}\left[\alpha\frac{1-q}{q}\theta - \frac{\alpha(1-\alpha)}{2}\left(\frac{1-q}{q}\theta\right)^2\right] \\
&= \alpha\theta - \frac{\alpha(1-\alpha)}{2}\frac{1-q}{q}\theta^2 \\
&= \alpha\theta\left(1 - \frac{(1-\alpha)(1-q)}{2q}\theta\right) := A
\end{aligned}$$

To prove $f(x(\theta)) > 0$ – i.e. (112) – it suffices to prove function $f(A) > 0$. To calculate $f(A)$, the following equation helps:

$$\begin{aligned}
\theta - A &= \theta - \alpha\theta + \alpha\theta\frac{(1-\alpha)(1-q)}{2q}\theta \\
&= \theta(1-\alpha)\left[1 + \alpha\frac{(1-q)}{2q}\theta\right].
\end{aligned}$$

With this equation,

$$\begin{aligned}
f(A) &= \frac{(1-q)(\theta - A)}{q + (1-q)\theta} - \frac{1}{2}\frac{\alpha}{1-\alpha}\left(\frac{(1-q)(\theta-A)}{q+(1-q)\theta}\right)^2 - \left(\frac{\alpha\theta}{A} - 1\right) \\
&= \frac{(1-q)\theta(1-\alpha)\left[1 + \alpha\frac{(1-q)}{2q}\theta\right]}{q + (1-q)\theta} - \frac{1}{2}\alpha(1-\alpha)\left(\frac{(1-q)}{q+(1-q)\theta}\right)^2\left(\theta\left[1 + \alpha\frac{(1-q)}{2q}\theta\right]\right)^2 \\
&\quad - \frac{\frac{(1-\alpha)(1-q)}{2q}\theta}{1 - \frac{(1-\alpha)(1-q)}{2q}\theta}.
\end{aligned}$$

Hence, $f(A) > 0 \Leftrightarrow$

$$\begin{aligned}
\frac{(1-q)\theta\left[1 + \alpha\frac{(1-q)}{2q}\theta\right]}{q + (1-q)\theta} - \frac{1}{2}\alpha\left(\frac{(1-q)}{q+(1-q)\theta}\right)^2\left(\theta\left[1 + \alpha\frac{(1-q)}{2q}\theta\right]\right)^2 - \frac{\frac{(1-q)}{2q}\theta}{1 - \frac{(1-\alpha)(1-q)}{2q}\theta} &> 0 \Leftrightarrow \\
\frac{1 + \alpha\frac{(1-q)}{2q}\theta}{q + (1-q)\theta} - \frac{1}{2}(1-q)\theta\alpha\left(\frac{1}{q+(1-q)\theta}\right)^2\left(1 + \alpha\frac{(1-q)}{2q}\theta\right)^2 - \frac{\frac{1}{2q}}{1 - \frac{(1-\alpha)(1-q)}{2q}\theta} &> 0
\end{aligned} \tag{113}$$

Let

$$z := \frac{1 + \alpha \frac{(1-q)}{2q} \theta}{q + (1-q)\theta}.$$

Then $z > \frac{1}{q+(1-q)\theta} > 1$ and $z = \frac{1}{q} \frac{1 + \frac{(1-q)}{q} \frac{\alpha\theta}{2}}{1 + \frac{(1-q)}{q}\theta} < \frac{1}{q}$; and inequality (113) follows from $g(z) := -\frac{1}{2}\alpha(1-q)\theta z^2 + z - \frac{1}{2q-(1-\alpha)(1-q)\theta} > 0$ for $z \in \left(1, \frac{1}{q}\right)$. For $z < \frac{1}{q}$, $g' = 1 - \alpha(1-q)\theta z > 1 - \alpha\theta\frac{1-q}{q} > 1 - \alpha\theta$, where the last inequality uses the fact that $q > \frac{1}{2}$ due to Assumption 3. Hence, $g(z) > 0$ for $z \in \left(1, \frac{1}{q}\right)$ follows from $g(1) \geq 0 \Leftrightarrow$

$$1 - \frac{1}{2}\alpha(1-q)\theta - \frac{1}{2q - (1-\alpha)(1-q)\theta} \geq 0. \tag{114}$$

The left hand side of inequality (114) decreases with θ and we know $\theta \leq 1$ for $\sigma \geq \sigma_{c1}$. Hence, inequality (114) holds for any $\theta \leq 1$ if it holds from $\theta = 1$, that is,

$$1 - \frac{1}{2}\alpha(1-q) - \frac{1}{2q - (1-\alpha)(1-q)} \geq 0,$$

which is assumed in Assumption 3. ■

Proof of Lemma 7:

Proof. The Lagrangean of problem (13) is:

$$\begin{aligned} \mathcal{L} &= \mathbf{E}_s\left((h + d\gamma_s - h_s')p_s + \beta V(h_s')\right) + \\ &\quad \lambda \times (\chi h - d)\, p + (1-q)\,\mu \times (h + \gamma_b d - h_b' - M)p_b. \end{aligned}$$

The first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial h_g'} &= -p_g + \beta\left[p + \lambda\chi p + (1-q)\,\mu p_b\right] = 0 \\ \frac{\partial \mathcal{L}}{\partial h_b'} &= -p_b(1+\mu) + \beta\left[p + \lambda\chi p + (1-q)\,\mu p_b\right] = 0 \\ \frac{\partial \mathcal{L}}{\partial d} &= \mathbf{E}_s(\gamma_s p_s) - \lambda p + (1-q)\,\mu\gamma_b = 0. \end{aligned}$$

Observe that these three first-order conditions (FOCs) are exactly the same as conditions (76) - (78) in the proof of Lemma 2, namely, the FOCs of the bank's problem without any intervention. Given that the three results of Lemma 2 follow from these FOCs, they also hold with Policy 2. ■

Proof of Lemma 8:

Proof. By (42), $r_{P1} = \frac{r_f}{q} - \frac{1-q}{q}\phi\tau_{P1}$. Hence, $r'_{P1} = -\frac{1-q}{q}\left(\tau_{P1} + \phi\tau'_{P1}(\phi)\right) \leq -\frac{1-q}{q}\tau_{P1} < -\frac{1-q}{q} = r'_{P2}(\phi)$, where " $\leq$ " holds because $\tau'_{P1}(\phi) \leq 0$ always. ■

Proof of Lemma 9:

Proof. We find that

$$\widetilde{\phi}_d := \frac{r_f - q\left(\alpha\sigma\theta(\sigma) - 1\right)}{1 + q\left(\alpha\sigma\theta(\sigma) - 1\right)} = \frac{\tau_3^* - \theta(\sigma)}{\theta(\sigma) + \frac{1-q}{q\sigma\alpha}}, \tag{115}$$

where $\tau_3^*(\sigma)$ is the value of τ^* in Phase 3, given in (93). Because $\alpha\sigma\theta(\sigma) > 1$, $\widetilde{\phi}_d < r_f$. Moreover, if $\sigma \geq \sigma_{c2}$, because $\tau_3^*(\sigma) > \theta(\sigma)$ by Lemma A1, $\widetilde{\phi}_d > 0$. Together, $\widetilde{\phi}_d \in (0, r_f)$ if $\sigma \geq \sigma_{c2}$. ■

Proof of Proposition 3:

Proof. By (64) $\tau_{P2}(\phi)$ decreases over $\phi \in \left[0, \widetilde{\phi}_d\right]$ from $\tau_3^*(\sigma)$ to $\theta(\sigma)$, where $\tau_3^*(\sigma)$ is the value of τ^* in Phase 3, given in (93). This optimisation problem 71 is therefore equivalent to:

$$\tau_{P2}^* = \arg\max_{\tau\in\left[\theta(\sigma),\tau_3^*(\sigma)\right]} \eta(\tau,\sigma). \tag{116}$$

It is not difficult to find that given σ, function $\eta(\tau,\sigma)$ is in a "U" shape over $\tau \in (0,\infty)$ and reaches the bottom at $\tau = \frac{1}{\sqrt{\sigma\alpha}}$. Therefore, the solution to the maximisation problem (116) lies at one of the ends: $\tau_{P2}^* \in \{\theta(\sigma), \tau_3^*(\sigma)\}$. By (35) $\eta(\tau,\sigma) = \frac{1}{\alpha[q\sigma+(1-q)]}\left[q^2\sigma\alpha + (1-q)^2 + q(1-q)\left(\tau^{-1} + \sigma\alpha\tau\right)\right]$. Therefore, $\eta(\theta(\sigma),\sigma) > \eta(\tau_3^*(\sigma),\sigma)$ if and only if

$$\begin{aligned}
\theta^{-1} + \sigma\alpha\theta &> \tau_3^{*-1} + \sigma\alpha\tau_3^* \Leftrightarrow \\
\theta^{-1} - \tau_3^{*-1} &> \sigma\alpha\left(\tau_3^* - \theta\right)|_{\tau_3^*-\theta>0} \Leftrightarrow \\
\frac{1}{\theta} &> \sigma\alpha\tau_3^*,
\end{aligned}$$

which, as τ_3^* is given by (93), is equivalent to

$$\theta(\sigma) < \frac{q}{r_f + q}. \tag{117}$$

Because $\theta'(\sigma) < 0$ by (6), Inequality (117) is equivalent to $\sigma > \sigma_p$, where σ_p is determined by

$$\theta(\sigma_p) = \frac{q}{r_f + q}, \tag{118}$$

namely (73). Therefore,

$$\tau_{P2}^{*}=\begin{cases}\tau_{3}^{*}\left(\sigma\right), & \text{if } \sigma\in\left[\sigma_{c2},\sigma_{p}\right];\\ \theta\left(\sigma\right), & \text{if } \phi\in\left[\sigma_{p},\infty\right),\end{cases}\tag{119}$$

Because $\tau_{P2}\left(\phi\right)=\tau_{3}^{*}\Leftrightarrow\phi=0$ and $\tau_{P2}\left(\phi\right)=\theta\Leftrightarrow\phi=\widetilde{\phi}_{d}$, Equation (119) leads to Equation (72). At $\sigma=\sigma_{p}$ $e_{P2}\left(\widetilde{\phi}_{d}\right)=e_{P2}\left(0\right)$ because $\eta\left(\theta,\sigma\right)=\eta\left(\tau_{3}^{*},\sigma\right)$. Hence the first part of the proposition is proved.

For the second part, first, because $\theta\left(\sigma\right)>\frac{1}{\sigma\alpha}$, from (118) we have $\frac{q}{r_{f}+q}=\theta\left(\sigma_{p}\right)>\frac{1}{\sigma_{p}\alpha}\Leftrightarrow\sigma_{p}>\frac{1}{\alpha}\left(1+\frac{r_{f}}{q}\right)=\sigma_{c2}$. Second, $e_{P2}\left(\widetilde{\phi}_{d}\right)>\frac{1}{1+r_{f}}$ by (70). It follows from (97) that $\eta\left(\tau_{P2}\left(\widetilde{\phi}_{d}\right),\sigma\right)>1$ for any σ. By the definition of $\widetilde{\phi}_{d}$, $\tau_{P2}\left(\widetilde{\phi}_{d}\right)=\theta\left(\sigma\right)$. Hence, $\eta\left(\theta\left(\sigma\right),\sigma\right)>1$. However, $\eta\left(\tau_{3}^{*}\left(\sigma_{e}\right),\sigma_{e}\right)=1$ by (97). Altogether, $\eta\left(\theta\left(\sigma_{e}\right),\sigma_{e}\right)>\eta\left(\tau_{3}^{*}\left(\sigma_{e}\right),\sigma_{e}\right)\Leftrightarrow\sigma_{p}<\sigma_{e}$. Therefore, $\sigma_{p}\in\left(\sigma_{c2},\sigma_{e}\right)$. ■

Proof of Lemma 10:

Proof. It suffices to prove that $d\eta\left(\theta\left(\sigma\right),\sigma\right)/d\sigma<0$ and $\lim_{\sigma\to\infty}\eta\left(\theta\left(\sigma\right),\sigma\right)=1$. The second claim is straightforward, because $\lim_{\sigma\to\infty}\theta\left(\sigma\right)=0$ and $\lim_{\sigma\to\infty}\sigma\alpha\theta\left(\sigma\right)=1$; and hence, by (35),

$$\begin{aligned}\lim_{\sigma\to\infty}\eta\left(\theta\left(\sigma\right),\sigma\right) &= \lim_{\sigma\to\infty}\frac{\left[q+\left(1-q\right)\left(\sigma\alpha\theta\left(\sigma\right)\right)^{-1}\right]\left[q+\left(1-q\right)\theta\left(\sigma\right)\right]}{q+\left(1-q\right)\sigma^{-1}}\\ &= \frac{\left[q+\left(1-q\right)\right]\cdot q}{q}\\ &= 1.\end{aligned}$$

To prove the first claim, as in the proof of Lemma 6, we write $\eta\left(\theta\left(\sigma\right),\sigma\right)$ as a function of θ – denoted by $\eta\left(\theta\right)$ – and use $x=\sigma^{-1}$, which, as a function of θ, is given by (111). Then

$$\eta\left(\theta\right)=\frac{\left[q+\left(1-q\right)x\left(\theta\right)\alpha^{-1}\theta^{-1}\right]\left[q+\left(1-q\right)\theta\right]}{q+\left(1-q\right)x\left(\theta\right)};$$

and $d\eta\left(\theta\left(\sigma\right),\sigma\right)/d\sigma=\eta'\left(\theta\right)\theta'\left(\sigma\right)$. Because $\theta'\left(\sigma\right)<0$, the first claim is equivalent to $\eta'\left(\theta\right)>0$ for $\theta>0$. By $\left(111\right)$, $q+\left(1-q\right)x\left(\theta\right)=q\left(1+\frac{1-q}{q}\theta\right)^{\alpha}$. Therefore,

$$\begin{aligned}\eta\left(\theta\right) &= q\left[1+\frac{1}{\alpha}\frac{1-q}{q}x\left(\theta\right)\theta^{-1}\right]\left(1+\frac{1-q}{q}\theta\right)^{1-\alpha}\\ |_{(111)} &= q\left[1+\frac{1}{\alpha}\left[\left(1+\frac{1-q}{q}\theta\right)^{\alpha}-1\right]\theta^{-1}\right]\left(1+\frac{1-q}{q}\theta\right)^{1-\alpha}.\end{aligned}$$

Let $t := 1 + \frac{1-q}{q}\theta \Leftrightarrow \theta = \frac{q}{1-q}(t-1) := \theta(t)$ and let $f(t) := \frac{1}{q}\eta(\theta(t))$. Then

$$\begin{aligned} f(t) &= \left[1 + \frac{1-q}{q\alpha} \cdot \frac{t^{\alpha}-1}{t-1}\right] t^{1-\alpha} \\ &= t^{1-\alpha} + \frac{1-q}{q\alpha} \cdot \frac{t - t^{1-\alpha}}{t-1}; \end{aligned}$$

and $\eta'(\theta) > 0$ for $\theta > 0$ if and only if $f'(t) > 0$ for $t > 1$. We find $f' = (1-\alpha)t^{-\alpha} + \frac{1-q}{q\alpha} \cdot \frac{1}{(t-1)^2} g(t)$, where $g(t) := \alpha t^{1-\alpha} + (1-\alpha)t^{-\alpha} - 1$. Hence, $f'(t) > 0$ for $t > 1$ if $g(t) > 0$ for $t > 1$, which holds true because $g(1) = 0$ and $g' = \alpha(1-\alpha)t^{-\alpha} - \alpha(1-\alpha)t^{-\alpha-1} = \alpha(1-\alpha)t^{-\alpha}(1-t^{-1}) > 0$ for $t > 1$. ■