The Formation of Protest Networks in Latin America[☆]

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Abstract

We analyze protest participation as a binary choice model in a communication network to coordinate actions. Both, the formation of links in the network as well as the action choices and beliefs about the support in the population are endogenous. We provide a complete characterization of the equilibrium action choices, beliefs and networks, where agents choose their strategies (actions and links) according to a perturbed best response update process, and update their beliefs following DeGroot's rule. We show that a threshold exists in the linking cost and the conformity parameter such that all agents coordinate on the same action. Further, we find that the introduction of incomplete information via beliefs lowers the threshold (i.e. makes protests more likely). Moreover, we show how the theoretical model can be efficiently estimated using cross sectional data on agents' choices and their network of interactions from Twitter data for two large scale protests in Latin America. Our estimation results further show that both, the local peer effect and the global conformity effect, are significant in explaining protest participation.

Key words: collective action, networks, riots, protests
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1. Introduction

Online social media platforms like Twitter provide open environments where people interact and communicate with each other and use communication networks to facilitate collective action, e.g. riots or protests.[1] We study games of incomplete information in which each person, given his local knowledge from the contacts in the communication network and beliefs about the overall support in the population, decides whether to participate in a riot or not. Our approach emphasizes the role of social networks in the emergence of collective action. Networks underlie payoffs, by governing local peer interaction, they channel crucial information, e.g. on the extent of support in the population, and they are endogenous, i.e. they co-evolve with actions and beliefs. Our model sheds novel (theoretical) light on the following issues: How does a large population coordinate within a network on collective action? How do expectations form and adapt along the process in the network? What is the role of individual heterogeneity (e.g. in preferences) for network formation and action adjustment?

We bring our model to the data by relying on "big" Twitter datasets on Latin American social unrest. We perform a structural estimation of model parameters and use random/fixed effects to capture the unobserved component in the idiosyncratic preferences of the agents (for retaining the status quo vs. changing it). We jointly estimate (and disentangle) the local peer effect and the global conformity effect in the agents' participation decision, and show that both are significant. The existing literature considered only one of them, but did not allow for their joint influence, and their results thus might be biased. Moreover, we show that ignoring the endogeneity of network formation may bias the estimates of these two effects. Finally, we show that the structural parameter estimates are similar across the two protest movements considered in Mexico and Brazil illustrating the robustness of our findings.

Our research relates to various strands of literature. There exists an extensive literature on coordination games in networks. Fixed networks have been analyzed in Blume [1993], Brock and Durlauf [2001] and Morris [2000]. A few more recent studies also allow for endogenous networks, that co-evolve with actions, such as Jackson and Watts [2002], Goyal and Vega-Redondo [2005], Hsie et al. [2018] and Ehrhardt et al. [2008]. Another strand of the literature investigates learning in networks including DeMarzo et al. [2003], Jackson and Golub [2010] and Acemoglu et al. [2014]. Moreover, there exists a long tradition in analyzing collective action and threshold behavior going back to Granovetter [1978] and more recent contributions including Chwe [2000] and Barberà and Jackson [2016]. Our model integrates the above features into a single framework leading to: (i) a closed-form characterization of equilibrium paths and full comparative analysis, (ii) a likelihood to be used in structural estimation of the model, and (iii) a real world application for two protests in Latin America using large scale Twitter data. Related empirical studies on peer effects in protest participation, such as González [2016] and Enikolopov et al. [2016], do not separate the local peer effect from a global conformity effect in the agents' participation decision as we do here.

The paper is organized as follows: In Section 2 we introduce the benchmark model with complete information. The game-theoretic setup can be found in Section 2.1. Section 2.2 contains the law of motion of actions and link adjustment. Section 2.3 provides a characterization of the invariant distribution and the stochastically stable states. Section 3 introduces the belief-based model with learning and incomplete (local) information, payoffs are discussed in Section 3.1, the revised law of motion under beliefs can be found in Section 3.2, and Section 3.3 characterizes stochastic stability under belief formation. The date for the empirical application is described

[1] See for example Earl and Kimport [2011], González-Bailón et al. [2011], González-Bailón [2017] and Priante et al. [2018] for a general discussion and further evidence.

in Section 4. The estimation procedure is discussed in Section 5 and the estimation results are given in Section 5.4. Finally, Section 6 concludes. Extensions of the model can be found in Appendix A, an alternative estimation method is introduced in Appendix B and all proofs are relegated to Appendix C.

2. Benchmark Model with Complete Information

2.1. Payoffs and Potential

Let the strategy of each agent $i \in \mathcal{N} = \{1, \ldots, n\}$ be given by $s_i \in \{-1, +1\}$ indicating whether i wants to participate in the riot ($s_i = +1$) or not ($s_i = -1$).[2] Further let $\mathbf{s} = (s_1, \ldots, s_n)^\top \in \{-1, +1\}^n$ where $\#(\{-1, +1\}^n) = 2^n$. Then the payoff of agent i is given by [cf. Brock and Durlauf, 2001][3,4]

$$\pi_i(\mathbf{s}, G) = (1 - \theta - \rho)\gamma_i s_i + \theta \sum_{j=1}^{n} a_{ij} s_i s_j + \rho \sum_{j \neq i}^{n} s_j s_i - \kappa s_i - \zeta d_i, \tag{1}$$

where $\gamma_i \in \{-1, +1\}$ is some idiosyncratic preference for participating in the riot, $\sum_{j=1}^{n} a_{ij} s_i s_j$ is the total group strategy choices identical to i, $a_{ij} \in \{0, 1\}$ indicates whether i and j are connected, $\theta \in (0, 1)$ is a local conformity (homophily) parameter, and $\rho \in (0, 1)$ is a global (population) conformity parameter, and consequently the factor $1 - \theta - \rho$ is weighting the idiosyncratic preference versus the group preference, with $\theta + \rho \leq 1$, $\kappa \geq 0$ is an adoption cost for choosing action $s_i = +1$ (e.g. the opportunity cost of rioting), d_i is the number of links of i in the network $G \in \mathcal{G}^n$, and $\zeta > 0$ a fixed linking cost.[5] Regarding the payoff function introduced in Equation (1), Rosser [1999] states that:

> "In this sort of an economy [...considered by Brock and Durlauf, 2001] with interacting agents, gradual changes in the degree of interaction (or coordination) or gradual changes in the willingness of agents to change their attitudes (intensity of choice) can lead to discontinuous changes, in which suddenly agents will be moving together in some very different direction, as in the takeoff or crash of a speculative bubble or the emergence or disappearance of "animal spirits" or coordination in a Keynesian macro model. One can imagine applications to the cases of fads and information contagion and cascades, or revolutions arising from a brave individual speaking out, although such models have not yet been applied in these cases."

Next, observe that there exists a potential function associated with the payoff function

[2] The strategy space is similar to the Ising model and the spin-glass model with two possible spin states [cf. Grimmett, 2010; Reichl, 2004; Sherrington and Kirkpatrick, 1975].

[3] See also Blume et al. [2011]; Brock and Durlauf [2007] for additional discussion of this type of binary choice model with (exogenous) social interactions. Further, Krauth [2006] analyzes the model by Brock and Durlauf [2001] on a cycle. De Paula and Tang [2012] introduce incomplete information and Lee et al. [2014] generalize it to arbitrary, fixed network structures.

[4] Similar to Phan and Semeshenko [2008] we assume that the agents' idiosyncratic preferences are heterogeneous and deterministic. In contrast, Brock and Durlauf [2001] assume that they are heterogeneous and random.

[5] We could consider a more general cost function given by $\zeta_1 d_i - \zeta_2 \sum_{j=1}^{n} a_{ij}(1 - s_i s_j)$ that allows for linking costs to be lower between agents choosing the same strategy. The corresponding payoff function is given by $\pi_i(\mathbf{s}, G) = (1 - \theta - \rho)\gamma_i s_i + (\theta + \zeta_2) \sum_{j=1}^{n} a_{ij} s_i s_j + \rho \sum_{j=1}^{n} s_j s_i - \kappa s_i - \zeta_1 d_i$. This is the same functional form as in Equation (12) up to a rescaling of the parameters.

introduced above [Monderer and Shapley, 1996].[6,7]

Proposition 1. *The payoff function in Equation (1) admits a potential function*

$$\Phi(\mathbf{s}, G) = (1-\theta-\rho)\sum_{i=1}^{n}\gamma_i s_i + \frac{\theta}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}s_i s_j + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} s_i s_j - \kappa\sum_{i=1}^{n} s_i - m\zeta, \quad (2)$$

for both action and link adjustments, where m counts the number of links in the network G.

The potential function in Proposition 1 will be useful to characterize the stationary states of the stochastic process of network formation and action adjustments that we will introduce in the following section.

2.2. Network Formation and Action Adjustment

We endogenize the action choices and the network using a stochastic process akin to Hsie et al. [2018]. In this process the opportunities for change arrive as a Poisson process [cf. Sandholm, 2010]. To capture the fact that agents are uncertain about the behavior of their neighbors or the consequences (costs) of their actions, we introduce noise in this decision process [cf. e.g. Blume, 1993; Kandori et al., 1993].

Definition 1. *The evolution of the population of agents and the links between them is characterized by a sequence of states $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$, $\omega_t \in \Omega$, where each state $\boldsymbol{\omega}_t = (\mathbf{s}_t, G_t)$ consists of a vector of agents' actions $\mathbf{s}_t \in \{-1,+1\}^n$ and a network $G_t \in \mathcal{G}^n$. In a short time interval $[t, t+\Delta t)$, $t \in \mathbb{R}_+$, one of the following events happens:*

Action adjustment *At rate $\chi \geq 0$ an agent $i \in \mathcal{N}$ is selected at random and given a revision opportunity of its current action $s_{it} \in \{-1,+1\}$. When agent i receives such a revision opportunity, he evaluates the marginal payoff from changing its current action s_{it} to s_i'. The computation of marginal payoffs is perturbed by an additive i.i.d. shock ε_{it}, so that the probability that we observe a switch from action s_{it} to s_i' is given by*

$$\begin{aligned}
&\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (s_i', \mathbf{s}_{-it}, G_t)|\boldsymbol{\omega}_t = (s_{it}, \mathbf{s}_{-it}, G_t)\right) \\
&\qquad = \chi\mathbb{P}\left(\pi_i(s_i', \mathbf{s}_{-it}, G_t) - \pi_i(s_{it}, \mathbf{s}_{-it}, G_t) + \varepsilon_{it} > 0\right)\Delta t + o(\Delta t) \\
&\qquad\qquad = \chi\mathbb{P}\left(\Phi(s_i', \mathbf{s}_{-it}, G_t) - \Phi(s_{it}, \mathbf{s}_{-it}, G_t) + \varepsilon_{it} > 0\right)\Delta t + o(\Delta t).
\end{aligned}$$

where we have used the fact that $\pi_i(s_i', \mathbf{s}_{-it}, G_t) - \pi_i(q_{it}, \mathbf{s}_{-it}, G_t) = \Phi(s_i', \mathbf{s}_{-it}, G_t) - \Phi(s_{it}, \mathbf{s}_{-it}, G_t)$. In the following we will make a specific assumption on the distribution of the random shocks. In particular, we assume that these shocks are independent and

[6] Let $\mathcal{G}(\pi_1, \pi_2, \ldots, \pi_n)$ be a game in strategic form with a finite number of players. The set of players is $\mathcal{N} = (1, 2, \ldots, n)$, the set of strategies of player i is $\mathcal{S}_i$ and the payoff function of each player is $\pi_i : \mathcal{S} \to \mathbb{R}$, where $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_n$ is the set of strategy profiles. A function $\Phi(\cdot) : \mathcal{S} \to \mathbb{R}$ is an exact potential function for game $\mathcal{G}$, if for every $i \in \mathcal{N}$ and for every $s_{-i} \in \mathcal{S}_{-i}$ and $s_i', s_i'' \in \mathcal{S}_i$: $\Phi(s_i', s_{-i}) - \Phi(s_i'', s_{-i}) = \pi_i(s_i', s_{-i}) - \pi_i(s_i'', s_{-i})$.

[7] Appendix A.2 shows how the potential function can be generalized to account for directed links.

identically exponentially distributed with parameter $\eta \geq 0$. We then can write[8]

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (s_i', \mathbf{s}_{-it}, G_t) | \boldsymbol{\omega}_t = (s_i, \mathbf{s}_{-it}, G_t)\right) &= \chi\, \mathbb{P}\left(-\varepsilon_{it} < \Phi(s_i', \mathbf{s}_{-it}, G_t) - \Phi(s_{it}, \mathbf{s}_{-it}, G_t)\right) \Delta t + o(\Delta t) \\ &= \chi \frac{e^{\eta \Phi(s_i', \mathbf{s}_{-i}, G_t)}}{e^{\eta \Phi(s_i', \mathbf{s}_{-it}, G_t)} + e^{\eta \Phi(s_i, \mathbf{s}_{-it}, G_t)}} \Delta t + o(\Delta t),\end{aligned}$$

Link formation *With rate $\lambda \geq 0$ a pair of agents ij which is not already connected receives an opportunity to form a link. The formation of a link depends on the marginal payoff the agents receive from the link plus an additive pairwise i.i.d. error term $\varepsilon_{ij,t}$. The probability that link ij is created is then given by*

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{s}_t, G_t + ij) | \boldsymbol{\omega}_{t-1} = (\mathbf{s}, G_t)\right) &= \lambda\, \mathbb{P}\left(\{\pi_i(\mathbf{s}_t, G_t + ij) - \pi_i(\mathbf{s}_t, G_t) + \varepsilon_{ij,t} > 0\}\right. \\ &\quad \left.\cap \{\pi_j(\mathbf{s}_t, G_t + ij) - \pi_j(\mathbf{s}_t, G_t) + \varepsilon_{ij,t} > 0\}\right) \Delta t + o(\Delta t) \\ &= \lambda\, \mathbb{P}\left(\Phi(\mathbf{s}_t, G_t + ij) - \Phi(\mathbf{s}_t, G_t) + \varepsilon_{ij,t} > 0\right) \Delta t + o(\Delta t),\end{aligned}$$

where we have used the fact that $\pi_i(\mathbf{s}_t, G_t + ij) - \pi_i(\mathbf{s}_t, G_t) = \pi_j(\mathbf{s}_t, G_t + ij) - \pi_j(\mathbf{s}_t, G_t) = \Phi(\mathbf{s}_t, G_t + ij) - \Phi(\mathbf{s}_t, G_t)$. Assuming that the error term $\varepsilon_{ij,t}$ is independently logistically distributed, we obtain for the creation of the link ij

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{s}_t, G_t + ij) | \boldsymbol{\omega}_t = (\mathbf{s}_t, G_t)\right) &= \lambda\, \mathbb{P}\left(-\varepsilon_{ij,t} < \Phi(\mathbf{s}_t, G_t + ij) - \Phi(\mathbf{s}_t, G_t)\right) \Delta t + o(\Delta t) \\ &= \lambda \frac{e^{\eta \Phi(\mathbf{s}_t, G_t + ij)}}{e^{\eta \Phi(\mathbf{s}_t, G_t + ij)} + e^{\eta \Phi(\mathbf{s}_t, G_t)}} \Delta t + o(\Delta t). \qquad (3)\end{aligned}$$

Link removal *With rate $\xi \geq 0$ a pair of linked agents i, j receives an opportunity to terminate their connection. The link is removed if at least one agent finds this profitable. The marginal payoffs from removing the link ij are perturbed by an additive pairwise i.i.d. error term $\varepsilon_{ij,t}$. The probability that the link ij is removed is then given by*

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{s}_t, G_t - ij) | \boldsymbol{\omega}_t = (\mathbf{s}, G_t)\right) &= \xi\, \mathbb{P}\left(\{\pi_i(\mathbf{s}_t, G_t - ij) - \pi_i(\mathbf{s}_t, G_t) + \varepsilon_{ij,t} > 0\}\right. \\ &\quad \left.\cup \{\pi_j(\mathbf{s}_t, G_t - ij) - \pi_j(\mathbf{s}_t, G_t) + \varepsilon_{ij,t} > 0\}\right) \Delta t + o(\Delta t) \\ &= \xi\, \mathbb{P}\left(\Phi(\mathbf{s}_t, G_t - ij) - \Phi(\mathbf{s}_t, G_t) + \varepsilon_{ij,t} > 0\right) \Delta t + o(\Delta t),\end{aligned}$$

where we have used the fact that $\pi_i(\mathbf{s}_t, G_t - ij) - \pi_i(\mathbf{s}_t, G_t) = \pi_j(\mathbf{s}_t, G_t - ij) - \pi_j(\mathbf{s}_t, G_t) = \Phi(\mathbf{s}_t, G_t - ij) - \Phi(\mathbf{s}_t, G_t)$. When the error term is independently logistically distributed we obtain

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{s}_t, G_t - ij) | \boldsymbol{\omega}_t = (\mathbf{s}_t, G_t)\right) &= \xi\, \mathbb{P}\left(-\varepsilon_{ij,t} < \Phi(\mathbf{s}_t, G_t - ij) - \Phi(\mathbf{s}_t, G_t)\right) \Delta t + o(\Delta t) \\ &= \xi \frac{e^{\eta \Phi(\mathbf{s}_t, G_t - ij)}}{e^{\eta \Phi(\mathbf{s}_t, G_t - ij)} + e^{\eta \Phi(\mathbf{s}_t, G_t)}} \Delta t + o(\Delta t).\end{aligned}$$

The action adjustment process in Definition 1 incorporates the fact that agents tend to choose actions similar to their neighbors. This is supported, for example, by the empirical evidence presented in Magdy et al. [2016]. Analyzing a large online social communication network these authors find that users who are connected in the network share similar preferences,

[8] Let z be i.i. logistically distributed with mean 0 and scale parameter η, i.e. $F_z(x) = \frac{e^{\eta x}}{1+e^{\eta x}}$. Consider the random variable $\varepsilon = g(z) = -z$. Since g is monotonic decreasing, and z is a continuous random variable, the distribution of ε is given by $F_\varepsilon(y) = 1 - F_z(g^{-1}(y)) = \frac{e^{\eta y}}{1+e^{\eta y}}$.

and that network-based interactions serve as strong predictors of stance. The link formation and removal processes in Definition 1 incorporate the fact that agents tend to connect to other agents who choose similar actions. This is consistent with empirically observed online communication networks. For example, Borge-Holthoefer et al. [2015] find a tendency for like-minded people to connect to each other.

We can numerically implement the stochastic process introduced in Definition 1 using the "next reaction method" for simulating a continuous time Markov chain [cf. Anderson, 2012; Gibson and Bruck, 2000]. We will use this method throughout the paper to illustrate our theoretical predictions for various network statistics (see e.g. Figure 1).

Let $\mathcal{F}$ denote the smallest σ-algebra generated by $\boldsymbol{\sigma}(\boldsymbol{\omega}_t : t \in \mathbb{R}_+)$. The filtration is the non-decreasing family of sub-σ-algebras $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ on the measure space $(\Omega, \mathcal{F})$, with the property that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_t \subseteq \cdots \subseteq \mathcal{F}$. The probability space is given by the triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}\colon \mathcal{F} \to [0,1]$ is the probability measure satisfying $\int_\Omega \mathbb{P}(\boldsymbol{\omega})d\mu(\boldsymbol{\omega}) = 1$. As we will see below the sequence of states $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$, $\boldsymbol{\omega}_t \in \Omega$ induces an irreducible and positive recurrent (i.e. ergodic) time homogeneous Markov chain.

The one step transition probability matrix $\mathbf{P}(t) : \Omega^2 \to [0,1]$ from a state $\omega \in \Omega$ to a state $\omega' \in \Omega$ is given by $\mathbb{P}(\boldsymbol{\omega}_{t+\Delta t} = \boldsymbol{\omega}'|\mathcal{F}_t = \sigma(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_t = \boldsymbol{\omega})) = \mathbb{P}(\boldsymbol{\omega}_{t+\Delta t} = \boldsymbol{\omega}'|\boldsymbol{\omega}_t = \boldsymbol{\omega}) = q(\boldsymbol{\omega}, \boldsymbol{\omega}')\Delta t + o(\Delta t)$ if $\omega' \neq \omega$, where $q(\boldsymbol{\omega}, \boldsymbol{\omega}')$ is the transition rate from state $\boldsymbol{\omega}$ to state $\boldsymbol{\omega}'$. The transition rate matrix (or infinitesimal generator) $\mathbf{Q} = (q(\boldsymbol{\omega}, \boldsymbol{\omega}'))_{\boldsymbol{\omega},\boldsymbol{\omega}'\in\Omega}$ of the Markov chain is given by

$$q(\boldsymbol{\omega}, \boldsymbol{\omega}') = \begin{cases} \chi \frac{e^{\eta\Phi(s,\mathbf{s}_{-i},G)}}{e^{\eta\Phi(s,\mathbf{s}_{-i},G)} + e^{\eta\Phi(s',\mathbf{s}_{-i},G)}} & \text{if } \boldsymbol{\omega}' = (s_i', \mathbf{s}_{-i}, G) \text{ and } \boldsymbol{\omega} = (\mathbf{s}, G), \\ \lambda \frac{e^{\eta\Phi(\mathbf{s},G+ij)}}{e^{\eta\Phi(\mathbf{s},G+ij)} + e^{\eta\Phi(\mathbf{s},G)}} & \text{if } \boldsymbol{\omega}' = (\mathbf{s}, G + ij) \text{ and } \boldsymbol{\omega} = (\mathbf{s}, G), \\ \xi \frac{e^{\eta\Phi(\mathbf{s},G-ij)}}{e^{\eta\Phi(\mathbf{s},G-ij)} + e^{\eta\Phi(\mathbf{s},G)}} & \text{if } \boldsymbol{\omega}' = (\mathbf{s}, G - ij) \text{ and } \boldsymbol{\omega} = (\mathbf{s}, G), \\ -\sum_{\boldsymbol{\omega}'\neq\boldsymbol{\omega}} q(\boldsymbol{\omega}, \boldsymbol{\omega}') & \text{if } \boldsymbol{\omega}' = \boldsymbol{\omega}, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

satisfying the Chapman-Kolmogorov forward equation $\frac{d}{dt}\mathbf{P}(t) = \mathbf{P}(t)\mathbf{s}$ so that $\mathbf{P}(t) = \mathbf{I} + \mathbf{Q}\Delta t + o(\Delta t)$. As the Markov chain is time homogeneous, the transition rates are independent of time. The stationary distribution $\mu^\eta : \Omega \to [0,1]$ is then the solution to $\boldsymbol{\mu}^\eta\mathbf{P} = \boldsymbol{\mu}^\eta$, or equivalently $\boldsymbol{\mu}^\eta\mathbf{Q} = \mathbf{0}$ [cf. e.g. Norris, 1998].

2.3. Equilibrium Characterization

In the following proposition we completely characterize the equilibrium action choices and networks of the above stochastic process:

Proposition 2. *Consider a dynamic process $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$ in which agents' payoffs are randomly perturbed with additive i.i.d. logistically distributed shocks with parameter $\eta > 0$, and assume that agents choose their strategies according to a perturbed best response update rule as in Definition 1. Then this process induces an ergodic Markov chain with a unique stationary distribution μ^η defined on the measurable space $(\Omega, \mathcal{F})$ such that $\lim_{t\to\infty} \mathbb{P}(\boldsymbol{\omega}_t = (\mathbf{s}, G)|\boldsymbol{\omega}_0 = (\mathbf{s}_0, G_0)) = \mu^\eta(\mathbf{s}, G)$. The probability measure μ^η is given by*

$$\mu^\eta(\mathbf{s}, G) = \frac{e^{\eta\Phi(\mathbf{s},G)}}{\sum_{G'\in\mathcal{G}^n} \sum_{\mathbf{s}'\in\{-1,+1\}^n} e^{\eta\Phi(\mathbf{s}',G')}}. \tag{5}$$

Proposition 2 allows us to characterize the equilibria in a fully dynamic framework, where not only the strategies s_i but also the links a_{ij} are endogenous.

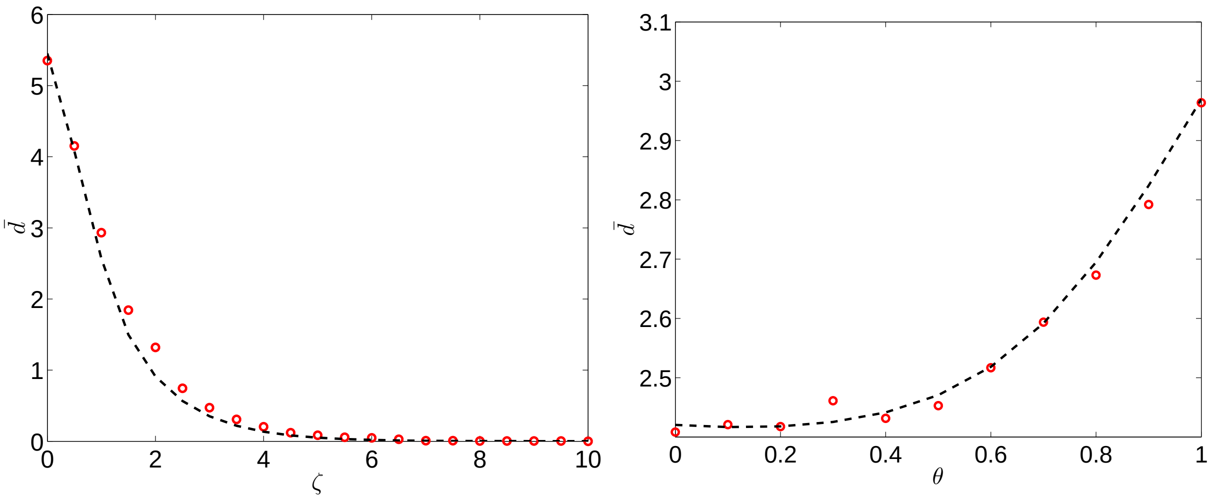

Figure 1: (Left panel) The average degree $\bar{d} = 2m/n$ across different values of the linking cost $\zeta \in \{0, 1, \ldots, 10\}$. The parameters used are $n = 10$, $n_+ = 5$, $\eta = 1$, $\lambda = \chi = \xi = 1$ and $\theta = 0.5$. (Right panel) The average degree $\bar{d}$ across different values of the conformity parameter $\theta \in [0, 1]$. The parameters used are $n = 10$, $n_+ = 5$, $\eta = 1$, $\lambda = \chi = \xi = 1$ and $\zeta = 1$. Dashed lines indicate the theoretical prediction of Equation (6) while circles indicate averages across 1000 numerical Monte Carlo simulations of the model.

The following proposition characterizes the expected number of links as a function of the parameters of the model.

Proposition 3. *The expected number of links in the stationary state is given by*

$$\mathbb{E}^{\eta}(m) = \frac{1}{\eta}\frac{1}{\mathscr{Z}^{\eta}}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}e^{\eta(1-\theta-\rho)(2k-n)}e^{\eta\left(\frac{\rho}{2}\left(n+2l(k,j)-\binom{n}{2}\right)\right)-\kappa(n-2(n_++k-2j))}$$
$$\times\left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}\left(\frac{l(k,j)}{1+e^{-\eta(\theta-\zeta)}}+\frac{\binom{n}{2}-l(k,j)}{1+e^{\eta(\theta+\zeta)}}\right), \quad (6)$$

where $l(k,j)$ is given by

$$l(k,j) = \frac{n^2 + (2(2j-k)-1)n + 2(2j-k)^2 - 2(n+2(2j-k)-n_+)n_+}{2}, \quad (7)$$

$n_+ = \#(\{\gamma_i = 1 : i = 1, \ldots, n\})$*, and we have that* $\lim_{\eta\to\infty}\mathbb{E}^{\eta}(m) = 0$.[9]

An example of the average degree $\bar{d} = 2m/n$ across different values of the linking cost $\zeta \in \{0, 1, \ldots, 10\}$ and the conformity parameter $\theta \in [0, 1]$ can be seen in Figure 1. As expected, the average degree is decreasing with increasing linking costs ζ and increasing with increasing conformity θ.

Next we consider the average action level. We can state the following proposition.

Proposition 4. *The expected average action level,* $\bar{s} = \frac{1}{n}\sum_{i=1}^{n} s_i$*, in the stationary state is*

[9] An explicit expression for the partition function $\mathscr{Z}^{\eta}$ can be found in Lemma 2 in Appendix C.

given by

$$\mathbb{E}^{\eta}(\bar{s}) = \frac{1}{\mathscr{Z}^{\eta}} \sum_{k=0}^{n} \sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j} \binom{n-n_+}{k-j} \frac{n+4j-2(n_++k)}{n} e^{\eta(1-\theta-\rho)(2k-n)}$$
$$\times\, e^{\eta(\frac{\rho}{2}(n+2l(k,j)-\binom{n}{2}))-\kappa(n-2(n_++k-2j))} \left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}} \left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}, \quad (8)$$

where $l(k,j)$ is defined in Equation (7) and $n_+ = \#(\{\gamma_i = 1 : i = 1, \dots, n\})$.

We conclude this section with a characterization of the the stationary state in the vanishing noise limit. When $\eta \to \infty$, the *stochastically stable* states in the support of μ^{η} are given by [Kandori et al., 1993]

$$\lim_{\eta\to\infty} \mu^{\eta}(\mathbf{s}, G) \begin{cases} > 0, & \text{if } \Phi(\mathbf{s}, G) \geq \Phi(\mathbf{s}', G'), \quad \forall \mathbf{s}' \in \{-1,+1\}^n, \quad G' \in \mathcal{G}^n, \\ = 0, & \text{otherwise.} \end{cases} \quad (9)$$

Note that the potential function of Equation (2) satisfies the "Single Crossing Differences" (SCD) property introduced in Arkolakis and Eckert [2017], and the authors show that this property is sufficient to guarantee that an iterative best response algorithm can find the global maximum of the potential.

The stochastically stable states are explicitly derived in the following proposition.

Proposition 5. *If $\theta < \zeta$ then the stochastically stable state in the limit of $\eta \to \infty$ is given by the empty network, $\overline{K_n}$ and*

1. *if $\rho > \rho^*$ then all agents choose the action $s_i = -1$,*
2. *if $\rho < \rho^*$ then all agents choose the action $s_i = \gamma_i$,*

where we have denoted by

$$\rho^* = \frac{1-\theta-\kappa}{n-n_++1}.$$

In the case of $\theta > \zeta$ the stochastically stable is either complete, K_n, or composed of two cliques, $K_{n^+} \cup K_{n-n^+}$, where in the first clique, K_{n^+}, all agents choose the actions $s_i = \gamma_i = +1$ while in the second clique, K_{n-n^+}, they choose the actions $s_i = \gamma_i = -1$. More precisely,

1. *if $n_+ < \frac{n}{2}$ and*
   - (a) *$\theta > \theta^*$ the stochastically stable state is the complete graph K_n in which all agents choose the action $s_i = -1$,*
   - (b) *$\theta < \theta^*$ the stochastically stable state is the union of two cliques, $K_{n_+} \cup K_{n-n_+}$, in which all agents choose the action $s_i = \gamma_i$,*
2. *if $n_+ > \frac{n}{2}$ and*
   - (a) *$\kappa > \kappa^*$ and*
     - i. *$\theta > \theta^*$ then the stochastically stable state is the complete graph K_n in which all agents choose the action $s_i = -1$,*
     - ii. *$\theta < \theta^*$ then the stochastically stable state is the union of two cliques, $K_{n_+} \cup K_{n-n_+}$, in which all agents choose the action $s_i = \gamma_i$,*
   - (b) *$\kappa < \kappa^*$ and*

i. $\theta > \theta^{**}$ *then the stochastically stable state is the complete graph* K_n *in which all agents choose the action* $s_i = +1$,

ii. $\theta < \theta^{**}$ *then the stochastically stable state is the union of two cliques,* $K_{n_+} \cup K_{n-n_+}$, *in which all agents choose the action* $s_i = \gamma_i$,

where $n_+ = \#(\{\gamma_i = 1 : i = 1, \ldots, n\})$ *and we have denoted by*

$$\begin{aligned}\theta^* &= \frac{(n-n_+)(\zeta-2\rho)+2(1-\kappa-\rho)}{2+n-n_+}, \\ \theta^{**} &= \frac{n_+(\zeta-2\rho)+2(1+\kappa-\rho)}{2+n_+}, \\ \kappa^* &= \frac{(1-\theta-\rho)(2n_+-n)}{n}.\end{aligned} \tag{10}$$

Figure 2 shows an illustration of the stochastically stable networks and action profiles in the (θ, κ)-parameter space in the case of $\theta > \zeta$ and $n_+ > n/2$.

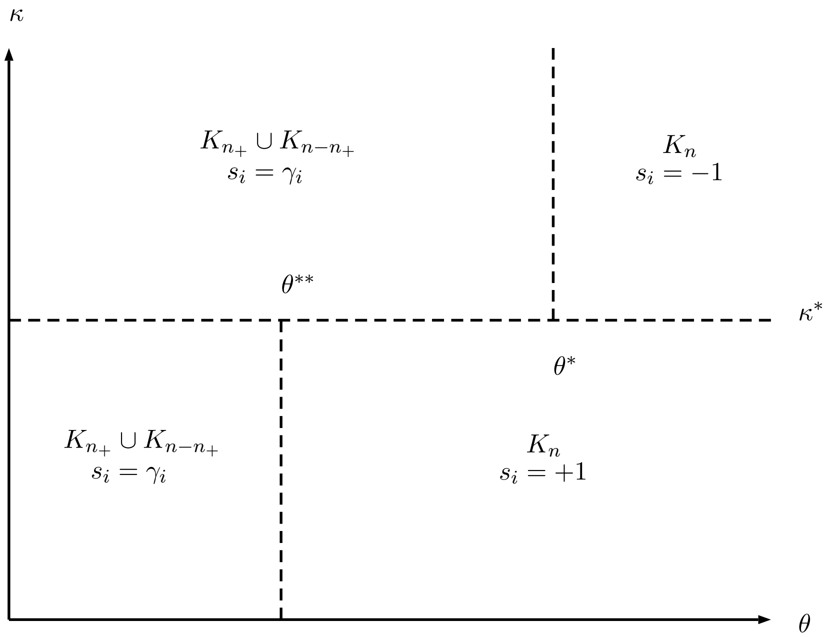

Figure 2: An illustration of the stochastically stable networks and action profiles in the (θ, κ)-parameter space in the case of $\theta > \zeta$ and $n_+ > n/2$.

Proposition 5 shows that when the idiosyncratic preference is large enough (i.e. θ is small enough) in the payoff function of Equation (1) then the society is segregated in disconnected communities in which each agent is choosing the action in accordance with her idiosyncratic preference ($\gamma_i = s_i$ for all $i = 1, \ldots, n$), while if the peer effect is strong enough (the conformity parameter θ is large enough) then the society becomes completely connected and all agents choose the same action. The action chosen in the latter case is determined by the idiosyncratic preference of the majority. That is, if more agents have an idiosyncratic preference $\gamma_i = +1$ (and $n_+ < n/2$) then all agents will chose $s_i = +1$, and vice versa. Finally, if linking is too costly ($\zeta > \theta$), then all agents are isolated and choose their idiosyncratic preference if the global conformity parameter ρ is not too high ($\rho < \rho^*$). For an illustration of the threshold in Equation (10) see Figure 3.

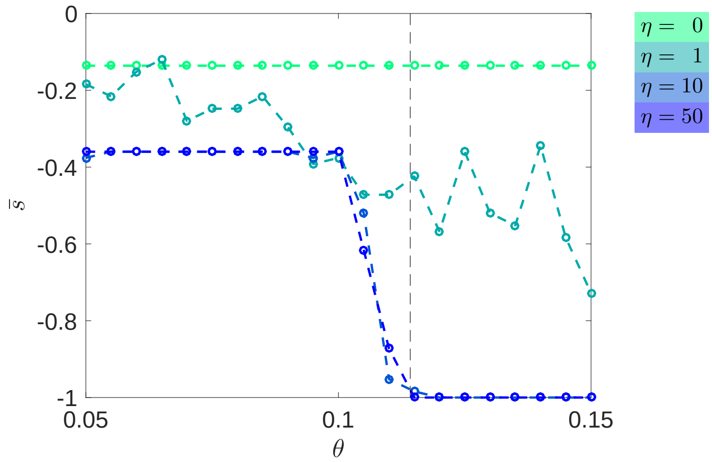

Figure 3: The average action level $\bar{s}$ indicated with circles from a simulation of the stochastic process of Definition 1 for varying values of θ with $n_+ = n/3$ using the "next reaction method" for simulating a continuous time Markov chain [cf. Anderson, 2012; Gibson and Bruck, 2000]. The threshold θ^* from Equation (10) is indicated with a vertical dashed line. For $\theta > \theta^*$ a homogeneous society appears in which all agents choose $s_i = -1$ while for $\theta < \theta^*$ the agents with idiosyncratic preference $\gamma_i = +1$ also choose $s_i = +1$ and a segregated society with two disconnected cliques of agents emerges.

We can analyze changes in the threshold in Equation (10) with respect to changes in the parameters:

$$\frac{\partial \theta^*}{\partial \rho} = \frac{2}{2 + n - n_+} - 2 < 0.$$

Hence, the threshold is decreasing in ρ, showing that the segregated equilibrium becomes harder to sustain the higher is the influence of the average action chosen in the population (global conformity effect).

From Equation (10) we can also derive the critical size n_+^* of the agents having an idiosyncratic preference $\gamma_i = +1$ so that all agents choose $s_i = +1$ if

$$n_+ > n_+^* = n - \frac{2(1 - \theta - \kappa - \rho)}{2\rho + \theta - \zeta}, \tag{11}$$

Figure 4 shows the number of agents choosing action $s_i = +1$ for varying values of n_+ ranging from 1 to n. The number of agents choosing action $s_i = +1$ exhibits a gradual transition from zero to n which is becoming increasingly sharp with increasing ρ.

3. Incomplete Information and Belief Formation

3.1. Payoffs, Potential and Beliefs

With the average action level, $\bar{s} = \frac{1}{n} \sum_{j \neq i}^{n} s_j$, we can write Equation (1) as

$$\pi_i(\mathbf{s}, G) = (1 - \theta - \rho)\gamma_i s_i + \theta \sum_{j=1}^{n} a_{ij} s_i s_j + \rho n \bar{s} s_i - \kappa s_i - \zeta d_i. \tag{12}$$

Instead of assuming complete knowledge about the population distribution of actions as in Section 2, in this section we assume that each agent i forms a belief $p_i \in [-1, +1]$ about the

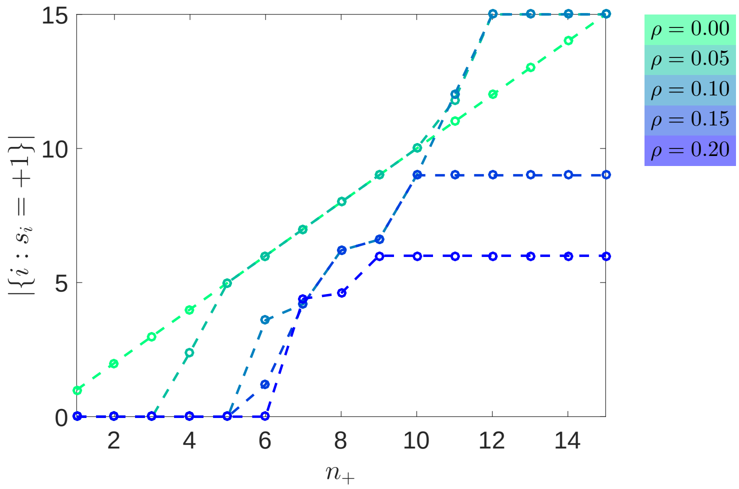

Figure 4: The number of agents choosing action $s_i = +1$ indicated with circles from a simulation of the stochastic process of Definition 1 for varying values of n_+ ranging from 1 to n using the "next reaction method" for simulating a continuous time Markov chain [cf. Anderson, 2012; Gibson and Bruck, 2000]. The number of agents choosing action $s_i = +1$ exhibits a gradual transition from 0 to n which is becoming increasingly sharp with increasing ρ.

average action level, $\bar{s}$, in the entire population. We then can modify Equation (12), as follows[10]

$$\mathbb{E}_i\left(\pi_i|\mathbf{s},\mathbf{p},G\right) = (1-\theta-\rho)\gamma_i s_i + \theta\sum_{j=1}^{n} a_{ij}s_is_j + \rho n p_i s_i - \kappa s_i - \zeta d_i, \tag{13}$$

where $\theta, \rho \in (0,1)$ and $\theta + \rho \leq 1$. The corresponding potential function is given by

$$\Phi(\mathbf{s},\mathbf{p},G) = (1-\theta-\rho)\sum_{i=1}^{n}\gamma_i s_i + \frac{\theta}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}s_is_j + \rho n\sum_{i=1}^{n} p_i s_i - \kappa\sum_{i=1}^{n} s_i - m\zeta, \tag{14}$$

where m denotes the number of links in G.

3.2. The Revised Law of Motion under Beliefs

In the following we consider a repeated process of action/network adjustment and belief updating. The action adjustment and link formation process is as in Definition 1, but with the potential function of Equation (14), taking the beliefs as given.

Definition 2. *The evolution of the population of agents and the links between them is characterized by a sequence of states $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$, $\omega_t \in \Omega$, where each state $\boldsymbol{\omega}_t = (\mathbf{s}_t, \mathbf{p}_t, G_t)$ consists of a vector of agents' actions, $\mathbf{s}_t \in \{-1,+1\}^n$, a vector of beliefs, $\mathbf{p}_t \in [-1,+1]^n$, and a network $G_t \in \mathcal{G}^n$. In a short time interval $[t, t+\Delta t)$, $t \in \mathbb{R}_+$, one of the following events happens:*

Action adjustment *At rate $\chi \geq 0$ an agent $i \in \mathcal{N}$ is selected at random and given a revision opportunity of its current action $s_{it} \in \{-1,+1\}$. When agent i receives such a revision opportunity, he evaluates the marginal payoff from changing its current action s_{it} to s_i'. The computation of marginal payoffs is perturbed by an additive i.i. logistically distributed shock*

[10] Related models for exogenous networks and a similar payoff function but with incomplete information have been studied in De Martí and Zenou [2014] and Lee et al. [2014]. Such an extension introduces a transition from a pure coordination game to a global game [cf. e.g. Angeletos et al., 2007; Angeletos and Pavan, 2007].

ε_{it}, so that the probability that we observe a switch from action s_{it} to s_i' is given by

$$\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (s_i', \mathbf{s}_{-it}, \mathbf{p}_t, G_t) | \boldsymbol{\omega}_t = (s_i, \mathbf{s}_{-it}, \mathbf{p}_t, G_t)\right) = \chi \frac{e^{\eta \Phi(s_i', \mathbf{s}_{-i}, \mathbf{p}_t, G_t)}}{e^{\eta \Phi(s_i', \mathbf{s}_{-it}, \mathbf{p}_t, G_t)} + e^{\eta \Phi(s_i, \mathbf{s}_{-it}, \mathbf{p}_t, G_t)}} \Delta t + o(\Delta t). \tag{15}$$

Link formation *With rate $\lambda \geq 0$ a pair of agetns ij which is not already connected receives an opportunity to form a link. The formation of a link depends on the marginal payoff the agents receive from the link plus an additive pairwise i.i. logistically distributed error term $\varepsilon_{ij,t}$. The probability that link ij is created is then given by*

$$\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{s}_t, \mathbf{p}_t, G_t + ij) | \boldsymbol{\omega}_t = (\mathbf{s}_t, \mathbf{p}_t, G_t)\right) = \lambda \frac{e^{\eta \Phi(\mathbf{s}_t, \mathbf{p}_t, G_t + ij)}}{e^{\eta \Phi(\mathbf{s}_t, \mathbf{p}_t, G_t + ij)} + e^{\eta \Phi(\mathbf{s}_t, \mathbf{p}_t, G_t)}} \Delta t + o(\Delta t). \tag{16}$$

Link removal *With rate $\xi \geq 0$ a pair of linked agents i, j receives an opportunity to terminate their connection. The link is removed if at least one agent finds this profitable. The marginal payoffs from removing the link ij are perturbed by an additive pairwise i.i. logistically distributed error term $\varepsilon_{ij,t}$. The probability that the link ij is removed is then given by*

$$\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{s}_t, \mathbf{p}_t, G_t - ij) | \boldsymbol{\omega}_t = (\mathbf{s}_t, \mathbf{p}_t, G_t)\right) = \xi \frac{e^{\eta \Phi(\mathbf{s}_t, \mathbf{p}_t, G_t - ij)}}{e^{\eta \Phi(\mathbf{s}_t, \mathbf{p}_t, G_t - ij)} + e^{\eta \Phi(\mathbf{s}_t, \mathbf{p}_t, G_t)}} \Delta t + o(\Delta t). \tag{17}$$

Belief updating *In a short time interval $[t, t + \Delta t)$, $t \in \mathbb{R}_+$, at rate $\tau \geq 0$ an agent $i \in \mathcal{N}$ is selected at random and updates his belief $p_{i,t} \in [-1, +1]$, according to a convex combination of the local average of the action choices and beliefs of his neighbors, that is*

$$p_{i,t+\Delta t} = \begin{cases} f_i(\mathbf{s}_t, \mathbf{p}_t, G_t) & \textit{with probability } \tau \Delta t, \\ p_{i,t} & \textit{with probability } 1 - \tau \Delta t, \end{cases}$$

where

$$f_i(\mathbf{s}_t, \mathbf{p}_t, G_t) = \varphi \underbrace{\frac{1}{d_{it}} \sum_{j=1}^{n} a_{ij,t} s_{jt}}_{\substack{\textit{local average} \\ \textit{actions}}} + (1 - \varphi) \underbrace{\frac{1}{d_{it}} \sum_{j=1}^{n} a_{ij,t} p_{jt}}_{\substack{\textit{local average} \\ \textit{beliefs}}}, \tag{18}$$

for $\varphi \in (0, 1]$.

The beliefs are updated in Equation (18) according to a convex combination of the local (i.e. across neighbors) average of the action choices and beliefs, as in the DeGroot model [cf. DeGroot, 1974].[11] Chandrasekhar et al. [2015]; Grimm and Mengel [2015] provide empirical evidence that individuals that attempt to learn the underlying state of the world in a network are best described by DeGroot models.

[11] See also Berger [1981]; DeMarzo et al. [2003]; Golub and Jackson [2012]; Jackson and Golub [2010].

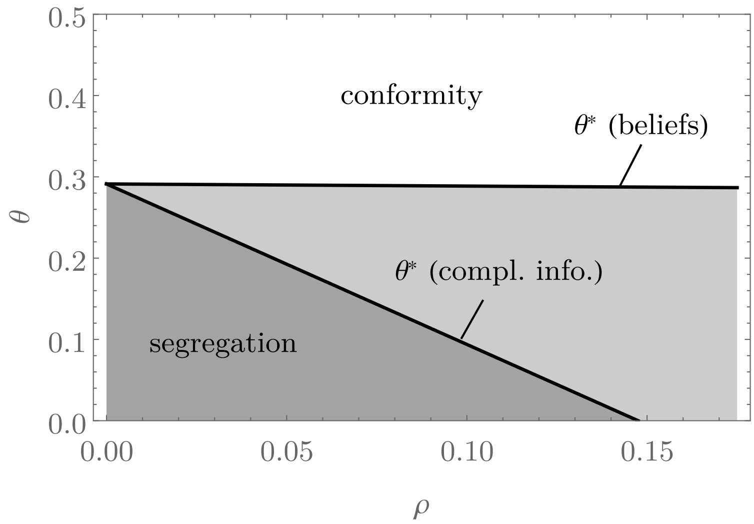

Figure 5: The threshold levels θ^* from Propositions 5 and 6, respectively. With incomplete information (and belief formation), the region where the disconnected cliques (segregation) are stochastically stable is larger. The light grey area indicates the parameter region where the introduction of incomplete information and DeGroot belief updating makes the segregated society more likely.

3.3. Stochastic Stability Under Belief Formation

The stochastically stable state of the stochastic process in Definition 2 is given in the following proposition.

Proposition 6. *In the stochastically stable state (in the limit of $\eta \to \infty$) we have that $p_i = s_i$ for all $i = 1, \ldots, n$ with the action profile given by $s_i = \gamma_i$ for all $i = 1, \ldots, n$ and when $\zeta < \theta$ a network composed of two cliques, $K_{n^+} \cup K_{n-n_+}$, of sizes n_+ and $n - n_+$, where in the first clique all agents choose the strategies $s_i = \gamma_i = +1$ and in the second clique they choose $s_i = \gamma_i = -1$ if the inequality*

$$\theta < \theta^* = \frac{(n - n_+)\zeta + 2(1 - \kappa - \rho)}{2 + n - n_+}, \tag{19}$$

holds, while if the inequality (19) is reversed and $\theta > 1 - (n+1)\rho + \kappa$ then the stochastically stable network is a complete graph K_n in which all agents $i = 1, \ldots, n$ choose $s_i = +1$ if $n_+ > \frac{n}{2}$ or $s_i = -1$ if $n_+ < \frac{n}{2}$ and $\theta > 1 - (n+1)\rho - \kappa$, and the network is empty when $\zeta > \theta$ and all agents choose their idiosyncratic preference.

From Proposition 6 we observe that the stochastically stable actions and network are the same as in Proposition 5, while the beliefs are identical to the actions. This implies that when the stochastically stable network is complete, then the beliefs (about the average action chosen in the entire population) are correct. But when the stochastically stable network is a union of two cliques, $K_{n^+} \cup K_{n-n_+}$, then the beliefs do not correspond to the the average action chosen in the entire population, but represent only the average action chosen in the local clique.

When comparing the stochastically stable states with complete information and with belief formation, we observe that the segregated equilibrium (the disconnected cliques) has a larger parameter region in which it is stable for the case with belief formation (cf. Proposition 6) than for the case with complete information (cf. Proposition 5). This is illustrated in Figure 8. In particular, the figure illustrates the parameter region where the introduction of incomplete information and the DeGroot belief updating makes the segregated society more likely.

A characterization of the stationary distribution for finite values of noise ($\eta < \infty$) can be obtained using a Laplace expansion around the stochastically stable state $\mathbf{s}^*$ (i.e. the potential maximizers) of Proposition 6 [cf. Wong, 2001, Theorem 3, p. 495].[12]

Figure 6 shows typical networks with increasing homophily for $\theta = 0.1$, $\theta = 0.125$, $\theta = 0.5$ and $\theta = 0.175$ from the top left panel to the bottom right panel using a Monte Carlo simulation of the model. With increasing θ the homogeneous society becomes more likely as predicted by Proposition 6. Moreover, for intermediate values of θ and the noise η two densely connected clusters with a few bridging ties between them emerge. This resembles real-world online communication networks [Priante et al., 2018].

Figure 7 shows the number of agents choosing action $s_i = +1$ with complete information (left panel) and incomplete information (right panel) indicated with circles from a simulation of the stochastic process of Definitions 1 and 2, respectively, for varying values of n_+ ranging from 1 to n. The transition to all agents choosing action $s_i = +1$ is becoming sharper in the case of incomplete information.

Finally, from Equation (19) we can derive the critical size n_+^* of the agents having an idiosyncratic preference $\gamma_i = +1$ so that all agents choose $s_i = +1$ if

$$n_+ > n_+^* = n - \frac{2(\theta + \kappa + \rho - 1)}{\zeta - \theta}. \tag{20}$$

Figure 8 shows the relative critical size n_+^*/n of the agents having an idiosyncratic preference $\gamma_i = +1$ so that all agents choose $s_i = +1$ with complete information from Equation (11) and beliefs from Equation (20). We observe that with beliefs, n_+^* is lower than with complete information. This indicates that the introduction of incomplete information and local belief formation makes riots more likely. Moreover, n_+^* is increasing with the peer conformity parameter θ.

[12] Consider the integral $J(\vartheta) = \int_{\mathcal{Q}^n} g(\mathbf{q}) e^{\vartheta f(\mathbf{q})} d\mathbf{q}$. Assume that (i) $J(\vartheta)$ is absolutely convergent for some $\vartheta > \vartheta_0$, (ii) f has a unique maximum at $\mathbf{q}_0$, and (iii) the Hessian $\left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_{\mathbf{q}=\mathbf{q}_0}$ is negative definite. Then $J(\vartheta) \sim \left(\frac{2\pi}{\vartheta}\right)^{\frac{n}{2}} g(\mathbf{q}_0) \left| \left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_{\mathbf{q}=\mathbf{q}_0} \right|^{-\frac{1}{2}} e^{-\vartheta f(\mathbf{q}_0)}$ for $\vartheta \to \infty$. For the case of multiple maxima and boundary solutions see Wong [2001].

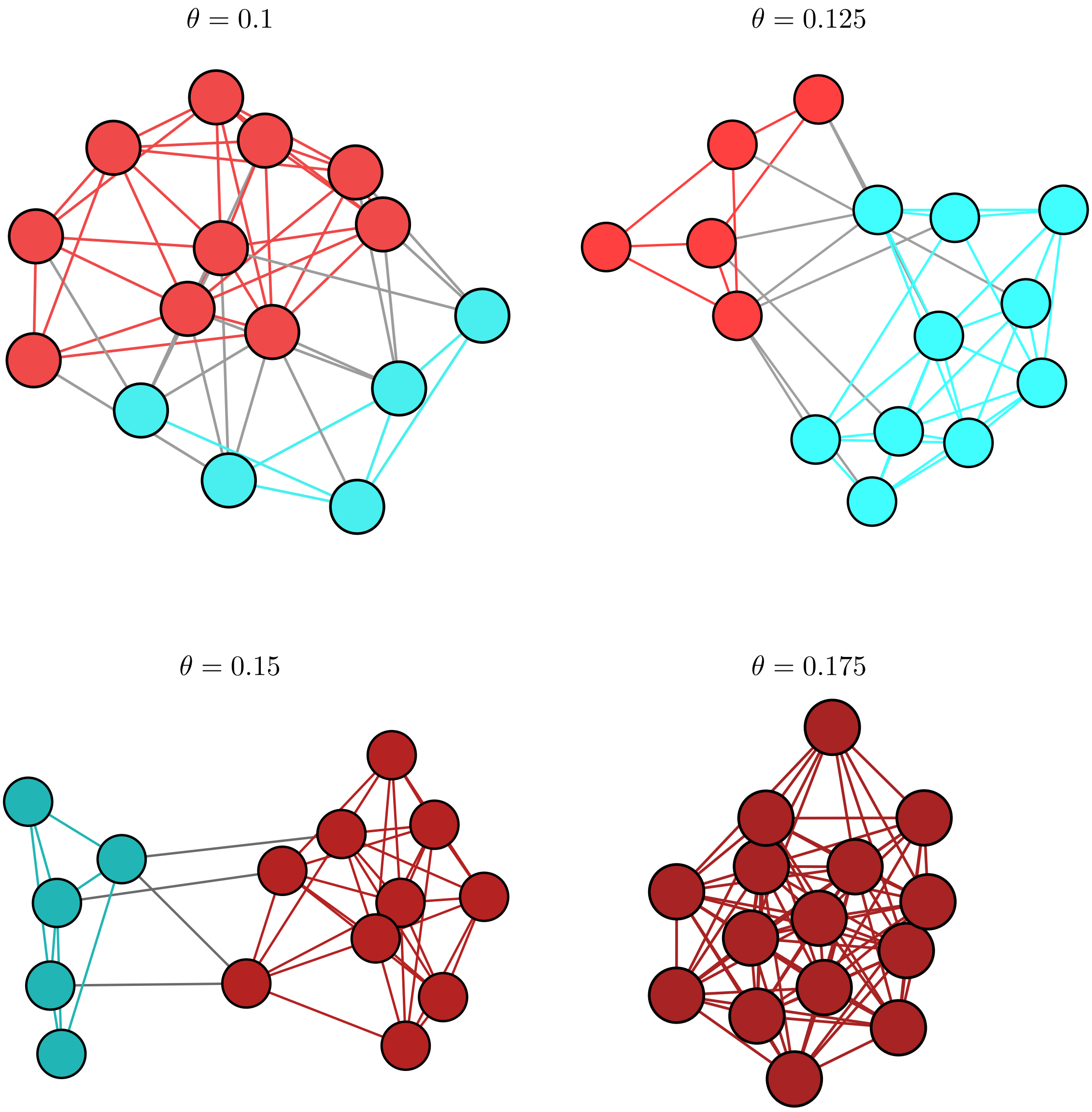

Figure 6: Typical networks with increasing homophily for $\theta = 0.1$, $\theta = 0.125$, $\theta = 0.5$ and $\theta = 0.175$ from the top left panel to the bottom right panel using a Monte Carlo simulation of the model. The nodes' colors indicate their action selected.

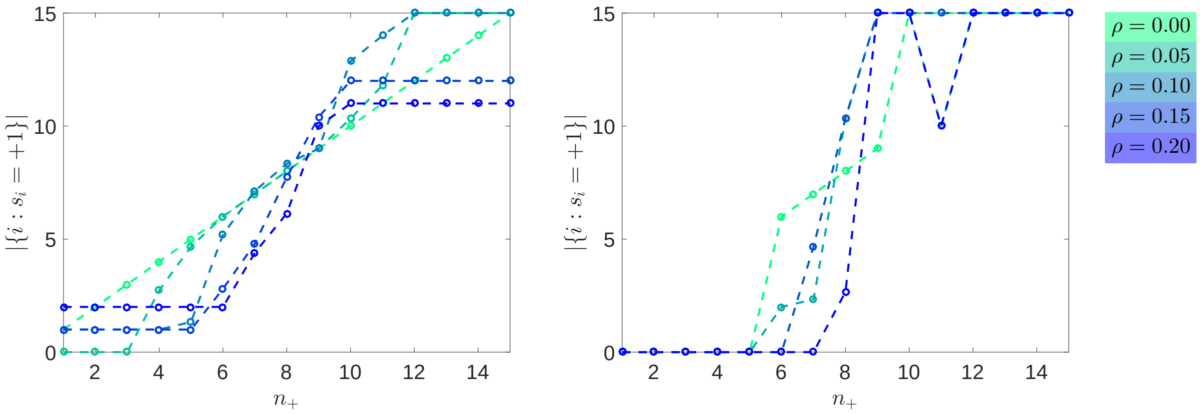

Figure 7: The number of agents choosing action $s_i = +1$ with complete information (left panel) and incomplete information (right panel) indicated with circles from a simulation of the stochastic process of Definitions 1 and 2, respectively, for varying values of n_+ ranging from 1 to n using the "next reaction method" for simulating a continuous time Markov chain [cf. Anderson, 2012; Gibson and Bruck, 2000]. The number of agents choosing action $s_i = +1$ exhibits a gradual transition from 0 to n which is becoming increasingly sharp with increasing ρ.

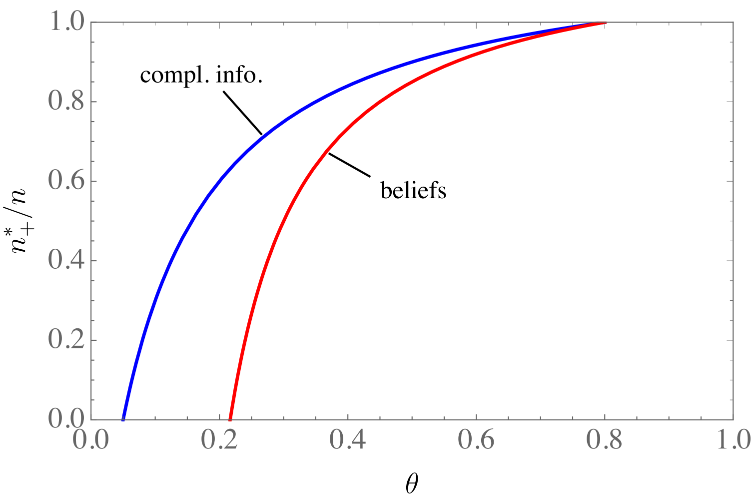

Figure 8: The relative critical size n^*_+/n of the agents having an idiosyncratic preference $\gamma_i = +1$ so that all agents choose $s_i = +1$ with complete information from Equation (11) and beliefs from Equation (20) as a function of θ. With beliefs, n^*_+ is lower than with complete information.

4. Data

Our ground truth dataset records the occurrences of civil-unrest in ten Latin American countries compiled for a research program [Ramakrishnan et al., 2014] by an independent group of social scientists and experts in Latin American politics. We have information about the exact location and the date of the event as well as the date when it was reported in a news source, event type (e.g., wages and employment; energy and resource), and population type such as labor and education. An overview of the volume of civil-unrest related events in Latin America can be seen in Figure 9.

For our empirical study we consider two specific protest movements: one from Mexico and the other from Brazil. The protest movement in Mexico is commonly referred to as "Yo Soy 132".[13] The sample period starts in 2012-5-31 and ends in 2012-06-30. We first gather the set of daily tweets from Mexico (using the country geolocation) in this period. We then filter out totally irrelevant "JustinBieber" type tweets. The action variable is $s_i = +1$ if "132" is used as a hashtag in the tweet content; otherwise $s_i = -1$. The network links are constructed among Twitter users by either retweeting or mentioning (and replying). We temporarily ignore the dynamic feature of the data and aggregate all actions and links into a static network. We further use a dictionary of 962 protest-related keywords obtained from political scientists and domain experts in Latin America for the same research program: *keywords* ("protesta", "revolucion", "marcha"), *key phrases* ("salir a la calle", "marcha por la paz"), and country-specific *key players* (political parties, heads of state, unions).

Table 1: Summary Statistics – Mexican sample

Mexican retweet links (sample size: 82,269)				
Variable	Mean	Std. Dev.	Min	Max
Action	-0.6711	0.7414	-1.0000	1.0000
Female	0.4660	0.2883	0.0500	0.9500
Capital	0.2538	0.4352	0.0000	1.0000
Log follower count	5.3168	1.5138	0.0000	15.4196
Kloutscore	31.5436	12.2596	10.0000	94.0000
Individual network degree	0.0159	0.1574	0.0000	12.0000
Mexican mentioning links (sample size: 60,837)				
Variable	Mean	Std. Dev.	Min	Max
Action	-0.8017	0.5977	-1.0000	1.0000
Female	0.4604	0.2710	0.0500	0.9500
Capital	0.2359	0.4245	0.0000	1.0000
Log follower count	5.7154	1.7495	0.0000	15.7113
Kloutscore	36.3754	12.1767	10.0000	98.0000
Individual network degree	0.1122	0.3583	0.0000	8.0000
Mexican retweet and mentioning links (sample size: 143,887)				
Variable	Mean	Std. Dev.	Min	Max
Action	-0.7778	0.6286	-1.0000	1.0000
Female	0.4672	0.2824	0.0500	0.9500
Capital	0.2396	0.4268	0.0000	1.0000
Log follower count	5.3394	1.5726	0.0000	15.7113
Kloutscore	32.3546	12.2151	10.0000	98.0000
Individual network degree	0.2048	0.5790	0.0000	25.0000

[13] The name Yo Soy 132, Spanish for "I Am 132", originated in an expression of solidarity with the original 131 protest's initiators. Twitter messages that referred to this event use the Hashtag #YoSoy132.

Country	Events
Mexico	4,454
Venezuela	3,072
Brazil	3,051
Paraguay	1,800
Argentina	1,227
Colombia	1,069
Chile	638
El Salvador	608
Uruguay	570
Ecuador	410

Figure 9: Events across countries in Latin America: November 1, 2012 - August 31, 2014. Previously appeared in Korkmaz et al. [2015, 2016].

Table 2: Comparison of whole and selected samples

	Mexico		Brazil	
Action	Whole sample	Selected sample	Whole sample	Selected sample
+1	25,124	13,529	173,654	34,712
-1	1,618,732	68,740	1,491,819	61,854
	1,643,856	82,269	1,665,473	96,566

Using Natural Language Processing (NLP), we obtain individual information on gender and location (with some missing values). We also obtain individual information on the number of followers, and their "Klout score".[14] Before any sample selection, we extract 1.64 million Twitter users, but most of them do not retweet or mention. Filtering out users who have less than two retweets (or mentions), we obtain 39,743 (30,837) users. We thus regard this as an initial sample for our empirical estimation. Summary statistics of our data can be found in Tables 1 and 2.

The second protest movement that we analyze is the chain of protests that took place in Brazil in June 2013, known as the Brazilian Spring. The respective Twitter sample starts in 2013-05-01 and ends in 2013-07-31, and includes tweets (in Portuguese, Spanish and English) from 85,789,395 users. The final sample is constructed in a similar way as the sample for the protests in Mexico (i.e., irrelevant tweets are filtered out). We compile a list of 56 hashtags relevant for the Brazilian Spring event (e.g., occupy brazil, brazil is awake) and identify the subset of individuals that use these hashtags in the period of study. The action variable is $s_i = +1$ if any of these hashtags is used in the tweet content; otherwise $s_i = -1$. Our final sample contains 277,805 users with $s_i = +1$ and their individual information (e.g., age, gender and Klout score). Summary statistics of the Brazilian sample can be found in Table 3.

Table 3: Summary Statistics – Brazilian retweet links

Variable	Mean	Std. Dev.	Min	Max
Action (± 1)	-0.2811	0.9597	-1.0000	1.0000
Female	0.5398	0.2885	0.0500	0.9500
Salo Paulo	0.0745	0.2626	0.0000	1.0000
Log follower count	6.5173	1.4345	0.0000	15.8226
Kloutscore	42.0628	7.9018	10.0000	92.0000
Individual network degree	0.1457	0.5714	0.0000	41.0000
Sample size	96,566			

5. Estimation

According to Proposition 1, the payoff function admits a potential function given by:[15]

$$\Phi(\mathbf{s}, G, \mathbf{\Theta}) = (1-\theta-\rho)\sum_{i=1}^{n}\gamma_i s_i + \frac{\theta}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} a_{ij}s_i s_j + \rho\sum_{i=1}^{n} n\bar{s}s_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} a_{ij}\zeta_{ij}. \tag{21}$$

Note that we have extended the payoff function in Equation (1) to allow for heterogeneous linking costs, $\zeta_{ij} = \zeta_{ji}$, across pairs of agents i and j, and this is captured by the term $\sum_{i=1}^{n}\sum_{j\neq i}^{n} a_{ij}\zeta_{ij}$ in Equation (21).[16] In the homogeneous case this reduces to $\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} a_{ij}\zeta = \frac{1}{2}\zeta\sum_{i=1}^{n} d_i = \zeta m$, where m denotes the number of links in G, as in Equation (2).

The link and action adjustment processes induce an ergodic Markov chain with a unique stationary distribution μ^η defined on the measurable space $(\Omega, \mathcal{F})$ such that $\lim_{t\to\infty}\mathbb{P}(\omega_t =$

[14] The Klout score measures the size of a user's social media network and correlates the activity and content posted to measure how other users interact with that content. Klout scores are a ranking that range between 0 to 100.

[15] Notice that the global conformity effect is captured by $\sum_{i=1}^{n} n\bar{s}s_i$ instead of $\sum_{i=1}^{n}\sum_{j\neq i}^{n} s_j s_i$, for preventing extra complexity in estimating the parameters of this potential function.

[16] See also Appendix A.1.

$(\mathbf{s}, G)|\omega_0 = (\mathbf{s}_0, G_0)) = \mu^\eta(\mathbf{s}, G)$. The probability measure $\mu^\eta(\mathbf{s}, G|\boldsymbol{\Theta})$ is given by

$$\mu^\eta(\mathbf{s}, G|\boldsymbol{\Theta}) = \frac{\exp(\eta\Phi(\mathbf{s}, G, \boldsymbol{\Theta}))}{\sum_{G' \in \mathcal{G}^n} \sum_{\mathbf{s}' \in \{-1,+1\}^n} \exp(\eta\Phi(\mathbf{s}', G', \boldsymbol{\Theta}))}. \tag{22}$$

In the following empirical analysis, we impose three parametric assumptions: first, since the error parameter η cannot be separately identified from other parameters, we follow the conventional exercise in the literature of discrete choice models to normalize it to one. Secondly, we replace individual heterogeneity $(1 - \theta - \rho)\gamma_i$ by an error function of $x_i\beta$, i.e., $\text{erf}(x_i\beta)$ with x_i denotes individual exogenous characteristics, to guarantee a bounded value between 1 and -1 in order to be consistent with the theoretical setting. Thirdly, we specify $\zeta_{ij} = h(x_i, x_j, \zeta)$ to capture heterogeneous linking cost. The specification of function h will be discussed later. The parameter vector $\boldsymbol{\Theta}$ is denoted by $\boldsymbol{\Theta} = (\theta, \rho, \beta^\top, \zeta^\top)^\top$.

We can specify latent variables in our empirical model to capture unobserved heterogeneity. Let $Z = (z_1, \cdots, z_n)^\top$ denote a vector of individual latent variables. When modeling individual heterogeneity by the error function $\text{erf}(x_i\beta)$, we can extend $x_i\beta$ to $x_i\beta + z_i\tau$ so that x_i and z_i stand for observed and unobserved individual characteristics, respectively. We can also introduce Z into the linking cost to reflect degree heterogeneity [Dzemski, 2014; Graham, 2017; Jochmans, 2018] by extending $h(x_i, x_j, \zeta)$ to $h(x_i, x_j, z_i, z_j, \zeta) = \zeta_0 + \sum_{k=1}^K |x_{ik} - x_{jk}|\zeta_k + z_i + z_j$.

In the following subsections, we discuss how to estimate $\boldsymbol{\Theta}$ from Equation (22). To motivate the estimation approach, we first consider two restrictive scenarios where the network is regarded exogenously given in the first scenario (Section 5.1) and the action profile is regard exogenously given in the second scenario (Section 5.2). Then we will discuss the joint estimation of network and action formation in Section 5.3.

5.1. Action Formation Conditional on the Network

We first consider the estimation of the model with an exogenously given network. According to Equation (22), the conditional probability of action profile on a given network is

$$\begin{aligned}
\mu(\mathbf{s}|G, \boldsymbol{\Theta}) &= \frac{\exp\left(\sum_{i=1}^n (\text{erf}(x_i\beta) + \rho n\bar{s})s_i + \frac{\theta}{2}\sum_{i=1}^n \sum_{j\neq i}^n a_{ij}s_is_j\right)}{\mathscr{Z}(G, \boldsymbol{\Theta})} \\
&= \frac{\exp\left(\mathbf{s}^\top(\text{erf}\,(\mathbf{X}\beta) + \rho n\bar{s}\cdot\mathbf{u}) + \frac{\theta}{2}\mathbf{s}^\top\mathbf{A}\mathbf{s}\right)}{\mathscr{Z}(G, \boldsymbol{\Theta})},
\end{aligned} \tag{23}$$

where $\mathscr{Z}(G, \boldsymbol{\Theta}) = \sum_{s' \in \{-1,+1\}^n} \exp\left(s'^\top(\text{erf}\,(\mathbf{X}\beta) + \rho n\bar{s}\cdot\mathbf{u}) + \frac{\theta}{2}s'^\top\mathbf{A}s'\right)$ is an intractable normalizing term and $\mathbf{u}$ is a vector of ones.

The model in Equation (23) is known as an Ising model [Ising, 1925] (or more generally an autologistic model by Besag [1974]). Estimating the autologistic model is difficult because the likelihood function of Equation (23) involves an intractable normalizing term and therefore a direct application of the maximum likelihood (ML) method is not computationally feasible when the network size is large. To deal with the estimation challenge, simulation-based methods are proposed to approximate the intractable normalizing term, such as the Markov chain Monte Carlo (MCMC) method by Geyer and Thompson [1992], Geyer [1994], and Gu and Zhu [2001].

A few other papers, including Sherman et al. [2006], Zheng and Zhu [2008], and Hughes et al. [2011], propose to use the Bayesian approach to estimate this model. Also note that the conventional Bayesian MCMC approach such as the Metropolis-Hastings (M-H) algorithm cannot be directly applied to estimate the autologistic model in Equation (23). Suppose we propose a new draw of $\boldsymbol{\Theta}$, $\widetilde{\boldsymbol{\Theta}}$, from the proposal distribution $q(\widetilde{\boldsymbol{\Theta}}|\boldsymbol{\Theta})$. The M-H algorithm requires us to calculate the acceptance probability

$$\alpha_{\boldsymbol{\Theta},MH}(\widetilde{\boldsymbol{\Theta}},\boldsymbol{\Theta}) = \min\left\{1, \frac{P(\mathbf{s}|\widetilde{\boldsymbol{\Theta}})P(\widetilde{\boldsymbol{\Theta}})q(\boldsymbol{\Theta}|\widetilde{\boldsymbol{\Theta}})}{P(\mathbf{s}|\boldsymbol{\Theta})P(\boldsymbol{\Theta})q(\widetilde{\boldsymbol{\Theta}}|\boldsymbol{\Theta})}\right\}. \tag{24}$$

However, $\alpha_{\boldsymbol{\Theta},MH}(\widetilde{\boldsymbol{\Theta}},\boldsymbol{\Theta})$ in Equation(24) is not calculable because there are intractable terms $z(\widetilde{\boldsymbol{\Theta}})$ and $z(\boldsymbol{\Theta})$ in the ratio and they do not cancel each other.

To overcome this problem, Murray et al. [2006] and Møller et al. [2006] propose the exchange algorithm which changes the acceptance probability to

$$\alpha_{\boldsymbol{\Theta},EX}(\widetilde{\mathbf{s}},\widetilde{\boldsymbol{\Theta}},\boldsymbol{\Theta}) = \min\left\{1, \frac{P(\mathbf{s}|\widetilde{\boldsymbol{\Theta}})P(\widetilde{\boldsymbol{\Theta}})q(\boldsymbol{\Theta}|\widetilde{\boldsymbol{\Theta}})}{P(\mathbf{s}|\boldsymbol{\Theta})P(\boldsymbol{\Theta})q(\widetilde{\boldsymbol{\Theta}}|\boldsymbol{\Theta})} \cdot \frac{P(\widetilde{\mathbf{s}}|\boldsymbol{\Theta})}{P(\widetilde{\mathbf{s}}|\widetilde{\boldsymbol{\Theta}})}\right\} \tag{25}$$

with auxiliary data $\widetilde{\mathbf{s}}$ simulated from $P(s|\widetilde{\boldsymbol{\Theta}})$ by the perfect sampling [Propp and Wilson, 1996]. Notice that the acceptance probability $\alpha_{\boldsymbol{\Theta},EX}(\widetilde{\mathbf{s}},\widetilde{\boldsymbol{\Theta}},\boldsymbol{\Theta})$ in Equation (25) can be calculated because all intractable terms are canceled. However, the perfect sampling is very costly and it is nearly impossible to apply to the autologistic model when the network size is large. To solve this computational problem, Liang [2010] proposes the double M-H algorithm to replace the perfect sampling. The idea of double M-H algorithm is to simulate auxiliary data $\widetilde{\mathbf{s}}$ by the standard M-H algorithm initiated from the observed data. An overview of the double M-H algorithm is summarized as follows:

Algorithm 1 (Double M-H Algorithm)**.** *At each iteration,*

Step I. *propose* $\widetilde{\boldsymbol{\Theta}}$ *from the proposal distribution* $q(\widetilde{\boldsymbol{\Theta}}|\boldsymbol{\Theta})$*.*

Step II. *simulate auxiliary data* $\widetilde{\mathbf{s}}$ *from* $P(s|\widetilde{\boldsymbol{\Theta}})$ *with* R *steps of the M-H algorithm initiated from the observed* $\mathbf{s}$*.*[17]

Step III. *update* $\boldsymbol{\Theta}$ *to the new draw* $\widetilde{\boldsymbol{\Theta}}$ *with the probability in Equation (25), otherwise, keep the original* $\boldsymbol{\Theta}$*.*

Practically, estimating an autologistic model for a large network data with the simulation approach which approximates the intractable normalizing term or with the Bayesian double M-H algorithm are still very time consuming. The most feasible approach to estimate an autologistic model turns out to be the maximum pseudo likelihood (MPL) method proposed by Besag [1974] and Besag [1975]. There are a few papers [Bhattacharya et al., 2018; Chatterjee et al., 2007; Comets, 1992; Guyon and Künsch, 1992] showing that the MPL estimator is consistent but not as efficient as the ML method. Given Equation (23), the conditional probability of agent i choosing action $s_i = 1$ given the action profiles of all other agents $\mathbf{s}_{-i}$ and the network G takes

[17] Notice that when simulating auxiliary data $\widetilde{\mathbf{s}}$ from $P(s|\widetilde{\boldsymbol{\Theta}})$, there is no issue of intractable normalizing term because in the acceptance probability of M-H, both numerator and denominator are evaluated under $\widetilde{\boldsymbol{\Theta}}$ and therefore can cancel out the intractable term. R is not necessary to be a big number. However, larger R may improve the performance of the algorithm.

the form:

$$\begin{aligned}\mu(s_i=1|\mathbf{s}_{-i},G,\mathbf{\Theta}) &= \frac{\mu(s_i=1,\mathbf{s}_{-i}|G,\mathbf{\Theta})}{\mu(s_i=1,\mathbf{s}_{-i}|G,\mathbf{\Theta})+\mu(s_i=-1,\mathbf{s}_{-i}|G,\mathbf{\Theta})}\\ &= \frac{\exp(\mathrm{erf}(x_i\beta)+\rho n\bar{s}+\theta\sum_{j\neq i}^{n}a_{ij}s_j)}{\exp(\mathrm{erf}(x_i\beta)+\rho n\bar{s}+\theta\sum_{j\neq i}^{n}a_{ij}s_j)+\exp(-\mathrm{erf}(x_i\beta)-\rho n\bar{s}-\theta\sum_{j\neq i}^{n}a_{ij}s_j)}\\ &= \frac{\exp(\mathrm{erf}(x_i\beta)+\rho n\bar{s}+\theta\sum_{j\neq i}^{n}a_{ij}s_j)}{2\cosh(\mathrm{erf}(x_i\beta)+\rho n\bar{s}+\theta\sum_{j\neq i}^{n}a_{ij}s_j)}. \end{aligned} \tag{26}$$

Similarly, the conditional probability of agent i choosing action $s_i=-1$ is

$$\mu(s_i=-1|\mathbf{s}_{-i},G,\mathbf{\Theta}) = \frac{\exp(-\mathrm{erf}(x_i\beta)-\rho n\bar{s}-\theta\sum_{j\neq i}^{n}a_{ij}s_j)}{2\cosh(\mathrm{erf}(x_i\beta)+\rho n\bar{s}+\theta\sum_{j\neq i}^{n}a_{ij}s_j)}. \tag{27}$$

Thus, the pseudo likelihood function for action profile $\mathbf{s}$ is defined as

$$\mu_{pseudo}(\mathbf{s}|G,\mathbf{\Theta}) = \prod_{i=1}^{n}\mu(s_i|\mathbf{s}_{-i},G,\mathbf{\Theta}) = \prod_{i=1}^{n}\mu(s_i=1|\mathbf{s}_{-i},G,\mathbf{\Theta})^{\frac{s_i+1}{2}}\mu(s_i=-1|\mathbf{s}_{-i},G,\mathbf{\Theta})^{\frac{1-s_i}{2}}. \tag{28}$$

5.2. Network Formation Conditional on Actions

Following Equation (22), we can model the conditional distribution of the network on given actions,[18]

$$\mu(G|\mathbf{s},\mathbf{\Theta}) = \frac{\mu(\mathbf{s},G|\mathbf{\Theta})}{\mu(\mathbf{s}|\mathbf{\Theta})} = \prod_{i=1}^{n}\prod_{j>i}^{n}\frac{\exp(a_{ij}(\theta s_i s_j-\zeta_{ij}))}{1+\exp(\theta s_i s_j-\zeta_{ij})}. \tag{29}$$

We model the linking cost $\zeta_{ij}=h(x_i,x_j,\zeta)$. An example of $h(x_i,x_j,\zeta)$ is $\zeta_0+\sum_{k=1}^{K}|x_{ik}-x_{jk}|\zeta_k$.

Even the pairwise independence in the conditional distribution of network links of Equation (29) allows us to compute the joint likelihood in a convenient way, the required computation is still heavy when the network size is large. To further alleviate the computational burden, we apply the case-control approach of Raftery et al. [2012], which can reduce the computational cost from $O(n^2)$ to $O(n)$. with an intention to further introduce latent variable to capture unobserved individual heterogeneity for network formation later.

Taking the probability function in Equation (29), we consider the log-likelihood function

$$\ell(G|\mathbf{s},\mathbf{\Theta}) = \sum_{i=1}^{n}\ell_i(a_{i,\cdot}|\mathbf{s},\mathbf{\Theta}), \tag{30}$$

[18] See Lemma 2 in Appendix F for the detail on deriving Equation (29).

where

$$
\begin{aligned}
\ell_i(a_{i,\cdot}|\mathbf{s},\boldsymbol{\Theta}) &= \sum_{j>i}^{n} a_{ij}(\theta s_i s_j - \zeta_{ij}) - \ln(1+\exp(\theta s_i s_j - \zeta_{ij})) \\
&= \sum_{j>i, a_{ij}=1} (\theta s_i s_j - \zeta_{ij} - \ln(1+\exp(\theta s_i s_j - \zeta_{ij}))) + \sum_{j>i, a_{ij}=0} (-\ln(1+\exp(\theta s_i s_j - \zeta_{ij}))) \\
&= l_{i,1} + l_{i,0}.
\end{aligned}
$$

Since the network links are sparse, the quantity $\ell_{i,0}$ can be viewed as a population total statistics. This population total can be estimated by a random sample of the population,

$$
\widetilde{\ell}_{i,0} = \frac{n_{i,0}}{m_{i,0}} \sum_{r=1}^{m_{i,0}} (-\ln(1+\exp(\theta s_i s_r - \zeta_{ir}))), \tag{31}
$$

where $n_{i,0}$ is the total number of zero's in the i^{th} row of the upper triangle of matrix $\mathbf{A}$, and $m_{i,0}$ is the number of samples selected from zero entries in the i^{th} row of the upper triangle of matrix $\mathbf{A}$. $\widetilde{\ell}_{i,0}$ is an unbiased estimator of $\ell_{i,0}$ given the random samples. When the network size is large, we can choose a small $m_{i,0}$ to compute $\widetilde{\ell}_{i,0}$ and reduce the amount of computation.

5.3. Joint Estimation of Action Adjustment and Network Formation

Following Sections 5.1 and 5.2, we can combine the conditional likelihood functions of actions and network to from the pseudo-likelihood function of actions and network links.[19] That is,

$$
\begin{aligned}
\mu_{pseudo}(\mathbf{s}, G|\boldsymbol{\Theta}) &\equiv \mu(G|\mathbf{s},\boldsymbol{\Theta}) \cdot \mu_{pseudo}(\mathbf{s}|G,\boldsymbol{\Theta}) \\
&= \prod_{i=1}^{n}\prod_{j>i}^{n} \frac{\exp(a_{ij}(\theta s_i s_j - \zeta_{ij}))}{1+\exp(\theta s_i s_j - \zeta_{ij})} \cdot \prod_{i=1}^{n} \mu(s_i = 1|\mathbf{s}_{-i}, G, \boldsymbol{\Theta})^{\frac{s_i+1}{2}} \mu(s_i = -1|\mathbf{s}_{-i}, G, \boldsymbol{\Theta})^{\frac{1-s_i}{2}}.
\end{aligned} \tag{32}
$$

Based on Equation (22), the formal likelihood function of the network G and the action profile $\mathbf{s}$ can be partitioned into

$$
\mu(\mathbf{s}, G|\boldsymbol{\Theta}) = \mu(G|\mathbf{s},\boldsymbol{\Theta})\mu(\mathbf{s}|\boldsymbol{\Theta}) = \prod_{i=1}^{n}\prod_{j>i}^{n} \frac{\exp(a_{ij}(\theta s_i s_j - \zeta_{ij}))}{1+\exp(\theta s_i s_j - \zeta_{ij})} \cdot \frac{1}{\mathscr{Z}(\boldsymbol{\Theta})} \sum_{G\in\mathcal{G}^n} \exp(\Phi(\mathbf{s}, G, \boldsymbol{\Theta})). \tag{33}
$$

[19] An alternative estimation approach can be found in Appendix B.

The estimation challenge of Equation (33) is to compute the marginal likelihood function of action profile,

$$
\begin{aligned}
\mu(\mathbf{s}|\boldsymbol{\Theta}) &= \frac{1}{\mathscr{Z}(\boldsymbol{\Theta})} \sum_{G \in \mathcal{G}^n} \exp(\Phi(\mathbf{s}, G, \boldsymbol{\Theta})) \\
&= \frac{1}{\mathscr{Z}(\boldsymbol{\Theta})} \sum_{G \in \mathcal{G}^n} \exp\left(\sum_{i=1}^{n} \operatorname{erf}(x_i\beta + \rho n \bar{s}) s_i + \sum_{i=1}^{n} \sum_{j \neq i}^{n} a_{ij} \left(\frac{\theta}{2} s_i s_j - \zeta_{ij} \right) \right) \\
&= \frac{1}{\mathscr{Z}(\boldsymbol{\Theta})} \exp\left(\sum_{i=1}^{n} \operatorname{erf}(x_i\beta + \rho n \bar{s}) s_i \right) \sum_{G \in \mathcal{G}^n} \exp\left(\sum_{i=1}^{n} \sum_{j \neq i}^{n} a_{ij} \left(\frac{\theta}{2} s_i s_j - \zeta_{ij} \right) \right) \\
&= \frac{1}{\mathscr{Z}(\boldsymbol{\Theta})} \exp\left(\sum_{i=1}^{n} \operatorname{erf}(x_i\beta + \rho n \bar{s}) s_i \right) \prod_{i=1}^{n} \prod_{j \neq i}^{n} \left(1 + \exp\left(\frac{\theta}{2} s_i s_j - \zeta_{ij} \right) \right) \\
&= \frac{1}{\mathscr{Z}(\boldsymbol{\Theta})} \exp\left(\sum_{i=1}^{n} \left(\operatorname{erf}(x_i\beta) + \rho n \bar{s} \right) s_i + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \ln\left(1 + \exp\left(\frac{\theta}{2} s_i s_j - \zeta_{ij} \right) \right) \right) \\
&= \frac{1}{\mathscr{Z}(\boldsymbol{\Theta})} \exp\left(\mathscr{H}(\mathbf{s}, \boldsymbol{\Theta}) \right), \qquad (34)
\end{aligned}
$$

where

$$
\mathscr{Z}(\boldsymbol{\Theta}) = \sum_{\mathbf{s} \in \{-1,+1\}^n} \exp\left(\mathscr{H}(\mathbf{s}, \boldsymbol{\Theta}) \right).
$$

The marginal likelihood function $\mu(\mathbf{s}|\boldsymbol{\Theta})$ of Equation (54) formulate the process of action profiles like a special type of Ising model with complete network (which is called Curie-Weiss model), and we may try to apply the pseudo likelihood approach for estimation. From Equation (54), we can derive the conditional probability $\mu(s_i = 1|\mathbf{s}_{-i}, \boldsymbol{\Theta})$,

$$
\begin{aligned}
\mu(s_i = 1|\mathbf{s}_{-i}, \boldsymbol{\Theta}) &= \frac{\mu(s_i = 1, \mathbf{s}_{-i}|\boldsymbol{\Theta})}{\mu(\mathbf{s}_{-i}|\boldsymbol{\Theta})} \\
&= \frac{\mu(s_i = 1, \mathbf{s}_{-i}|\boldsymbol{\Theta})}{\mu(s_i = 1, \mathbf{s}_{-i}|\boldsymbol{\Theta}) + \mu(s_i = -1, \mathbf{s}_{-i}|\boldsymbol{\Theta})} \\
&= \frac{\exp(\mathscr{H}(s_i = 1, \mathbf{s}_{-i}, \boldsymbol{\Theta}))}{\exp(\mathscr{H}(s_i = 1, \mathbf{s}_{-i}, \boldsymbol{\Theta})) + \exp(\mathscr{H}(s_i = -1, \mathbf{s}_{-i}, \boldsymbol{\Theta}))}. \qquad (35)
\end{aligned}
$$

Notice that $\mathscr{H}(\mathbf{s})$ can be decomposed into

$$
\mathscr{H}(\mathbf{s}, \boldsymbol{\Theta}) = \mathscr{H}_i(s_i, \mathbf{s}_{-i}, \boldsymbol{\Theta}) + \sum_{\ell \neq i}^{n} \left(\operatorname{erf}(x_\ell\beta) + \rho n \bar{s} \right) s_\ell + \sum_{\ell \neq i} \sum_{j \neq i, \ell} \ln\left(1 + e^{\frac{1}{2}\theta s_j s_\ell - \zeta_{j\ell}} \right), \qquad (36)
$$

where

$$
\mathscr{H}_i(s_i, \mathbf{s}_{-i}, \boldsymbol{\Theta}) = \left(\operatorname{erf}(x_i\beta) + \rho n \bar{s} \right) s_i + 2 \sum_{j \neq i}^{n} \left(\ln\left(1 + e^{\frac{1}{2}\theta s_i s_j - \zeta_{ij}} \right) \right).
$$

Therefore, we can further simplify Equation (35) into

$$
\mu(s_i = 1|\mathbf{s}_{-i}, \boldsymbol{\Theta}) = \frac{\exp\left(\mathscr{H}_i(s_i = 1, \mathbf{s}_{-i}, \boldsymbol{\Theta}) \right)}{\exp\left(\mathscr{H}_i(s_i = 1, \mathbf{s}_{-i}, \boldsymbol{\Theta}) \right) + \exp\left(\mathscr{H}_i(s_i = -1, \mathbf{s}_{-i}, \boldsymbol{\Theta}) \right)}. \qquad (37)
$$

Therefore, the likelihood partition approach plus pseudo likelihood for action profile $\mathbf{s}$ choose the parameter $\mathbf{\Theta}$ to maximize

$$\begin{aligned}\mu_{pseudo}(G, \mathbf{s}|\mathbf{\Theta}) &= \mu(G|\mathbf{s}, \mathbf{\Theta}) \cdot \mu_{pseudo}(\mathbf{s}|\mathbf{\Theta}) \\ &= \prod_{i=1}^{n}\prod_{j>i}^{n} \frac{\exp(a_{ij}(\theta s_i s_j - \zeta_{ij}))}{1+\exp(\theta s_i s_j - \zeta_{ij})} \cdot \prod_{i=1}^{n} \mu(s_i = 1|\mathbf{s}_{-i})^{\frac{s_i+1}{2}} \mu(s_i = -1|\mathbf{s}_{-i})^{\frac{1-s_i}{2}}. \quad (38)\end{aligned}$$

Notice that the pseudo likelihood function of Equation (38) is different from the pseudo likelihood function of Equation (32). In Equation (32), the pseudo likelihood function for action profile is $\mu_{pseudo}(\mathbf{s}|G, \mathbf{\Theta})$, while in Equation (38) the pseudo likelihood function for action profile is $\mu_{pseudo}(\mathbf{s}|\mathbf{\Theta})$, which removes the network G from the condition.

5.4. Empirical Estimation Result

The estimation results for Mexico and Brazil can be found in Tables 4 and 5, respectively. In each table, we report the pseudo likelihood estimation results for the model of actions conditional on network in Column (I); the model of network conditional on actions in Column (II); and the joint model of actions and network in Column (III).

From the results for Mexico in Table 4, we observe that both the estimated local spillover effect (θ) and the global conformity (ρ) are positive and significant. Comparing Column (I) with Column (III), the estimated local spillover effect drops from 0.2991 in Column (I) to 0.2384 in Column (III) and the estimated global conformity effect drops from 4.87E-06 to 1.37E-06, showing that ignoring the endogeneity of network formation may bias the estimates of θ and ρ. For network formation, the estimation results from the linking cost function show that being both female lowers the linking cost between each pair, same as having similar numbers of Tweeter followers and Klout scores. However, if both residing in the capital city, the linking cost will be higher.

The results for Brazil in Table 5 is largely similar to the results in Table 4.

6. Conclusion

In this paper we have introduced a model of protest participation mitigated via networks where the network co-evolves with actions and beliefs. We provide a complete characterization of the equilibrium action choices, beliefs and networks, and show that a threshold exists in the linking cost and the conformity parameter such that all agents coordinate on the same action. We further find that the introduction of incomplete information via beliefs lowers the threshold. We then bring our model to the data by relying on large-scale Twitter data from Latin American on social unrests in Mexico and Brazil. We perform a structural estimation of the model's parameters and use random/fixed effects to capture the unobserved component in the idiosyncratic preferences of the agents (for retaining the status quo vs. changing it). We jointly estimate (and disentangle) the local peer effect and the global conformity effect in the agents' participation decision, and show that both are significant. Moreover, we show that ignoring endogeneity from network formation may bias the estimates of these two effects. Finally, we show that the structural parameter estimates are similar across the two protest movements considered in Mexico and Brazil, illustrating the robustness of our findings.

Table 4: Pseudo likelihood estimation of actions and network with unobserved heterogeneity (Mexican Retweet and Mentioning links)

	actions conditional on network (I)	network conditional on actions (II)	joint action and network (III)
Local spillover (θ)	0.2991***	0.2496***	0.2384***
	(0.0110)	(0.0153)	(0.0047)
Global conformity (ρ)	4.87E-06***		1.37E-06***
	(1.92E-07)		(8.170E-08)
Individual heterogeneity (γ)			
female	-0.2941***		-0.4052***
	(0.0143)		(0.0031)
capital	0.1824***		0.1816***
	(0.0086)		(0.0080)
follower count	-0.0194***		-0.0849***
	(0.0041)		(0.0024)
kloutscore	-0.0085***		-0.0270***
	(0.0005)		(0.0018)
latent variable			-0.3249***
			(0.0137)
Linking cost (ζ)			
constant		14.4864***	14.3352***
		(0.0400)	(0.0052)
$\lvert \text{female}_i - \text{female}_j \rvert$		0.3386***	0.3197***
		(0.0347)	(0.0127)
$I(\text{capital}_i = \text{capital}_j)$		-1.0690***	-1.0150***
		(0.0284)	(0.0033)
$\lvert \text{follower}_i - \text{follower}_j \rvert$		0.1567***	0.1418***
		(0.0094)	(0.0048)
$\lvert \text{kloutscore}_i - \text{kloutscore}_j \rvert$		0.0386***	0.0398***
		(0.0012)	(0.0011)
variance of latent variable		1.5175***	1.3574***
		(0.0374)	(0.0168)
Sample size		143,887	

Notes: The estimates reported in this table are computed from the posterior mean of MCMC draws. The values reported in parentheses are posterior standard deviations. ***, **, and * indicate that the highest density range does not cover zero at 99%, 95%, and 90% level.

Table 5: Pseudo-likelihood estimation of actions and network (Brazilian retweet links)

	actions conditional on network	network conditional on actions	joint action and network	
	(I)	(II)	(III)	(IV)
Local spillover (θ)	0.1850***	0.1671***	0.1826***	0.1771***
	(0.0089)	(0.0116)	(0.0051)	(0.0079)
Global conformity (ρ)	-2.08E-05***		-2.01E-05***	-2.02E-05***
	(3.41E-07)		(3.49E-07)	(2.39E-07)
Individual heterogeneity (γ)				
female	-1.0408***		-0.8223***	-0.8692***
	(0.0644)		(0.0408)	(0.0177)
capital	0.4104***		0.3822***	0.3940***
	(0.0228)		(0.0138)	(0.0115)
follower count	-0.2865***		-0.2939***	-0.2845***
	(0.0111)		(0.0098)	(0.0093)
kloutscore	0.0277***		0.0281***	0.0274***
	(0.0015)		(0.0012)	(0.0012)
Linking cost (ζ)				
constant		12.7474***	12.8553***	12.7966***
		(0.0231)	(0.0134)	(0.0141)
$\lvert \text{female}_i - \text{female}_j \rvert$		0.3688***	0.2734***	0.3044***
		(0.0419)	(0.0165)	(0.0291)
$I(\text{capital}_i = \text{capital}_j)$		-1.8217***	-1.6673***	-1.8829***
		(0.0608)	(0.0306)	(0.0287)
$\lvert \text{follower}_i - \text{follower}_j \rvert$		0.1971***	0.1913***	0.1680***
		(0.0125)	(0.0095)	(0.0073)
$\lvert \text{kloutscore}_i - \text{kloutscore}_j \rvert$		0.0486***	0.0417***	0.0503***
		(0.0023)	(0.0018)	(0.0020)
Sample size		96,566		

Notes: The estimates reported in this table are computed from the posterior mean of MCMC draws. The values reported in parentheses are posterior standard deviations. ***, **, and * indicate that the highest density range does not cover zero at 99%, 95%, and 90% level.

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Appendix

A. Extensions

A.1. Heterogeneous Linking Costs

Note that we can allow for a pair specific cost ζ_{ij} for a link between i and j so that the payoff function of agent i reads as

$$\pi_i(\mathbf{s}, G) = (1-\theta)\gamma_i s_i + \theta \sum_{j=1}^{n} a_{ij} s_i s_j - \sum_{j=1}^{n} a_{ij}\zeta_{ij}. \tag{39}$$

The potential function is then given by

$$\Phi(\mathbf{s}, G) = (1-\theta)\sum_{i=1}^{n}\gamma_i s_i + \frac{\theta}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} s_i s_j - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\zeta_{ij},$$

and the same results as in Proposition 2 hold.

A.2. Directed Links

It is possible to consider a directed network. In this case the potential function of Proposition 1 needs to be modified as follows

$$\Phi(\mathbf{s}, G) = (1-\theta-\rho)\sum_{i=1}^{n}\gamma_i s_i + \theta\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{2}a_{ij}a_{ji} + a_{ij}(1-a_{ji})\right) s_i s_j + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} s_i s_j - \kappa\sum_{i=1}^{n} s_i - m\zeta. \tag{40}$$

B. Belief Propagation Algorithm

The formal likelihood function of the network G and the action profile $\mathbf{s}$ can be written as

$$P(\mathbf{s}, G) = P(G|\mathbf{s})P(\mathbf{s}) = \prod_{i<j}^{n} \frac{\exp(a_{ij}(\theta s_i s_j - \zeta_{ij}))}{1 + \exp(\theta s_i s_j - \zeta_{ij})} \cdot \exp(\mathscr{H}(\mathbf{s}) - \ln \mathscr{Z}), \tag{41}$$

where $\mathscr{H}(\mathbf{s})$ and $\mathscr{Z}$ are given in below. Let us assume that $\gamma_i \in \mathbb{R}$. We first compute the *partition function*.[20] We have that

$$\begin{aligned}
\mathscr{Z}^\eta &\equiv \sum_{G\in\mathcal{G}^n} \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\Phi(\mathbf{s},G)} \\
&= \sum_{\mathbf{s}\in\{-1,+1\}^n} \sum_{G\in\mathcal{G}^n} e^{\eta\left(\sum_{i=1}^n((1-\theta-\rho)\gamma_i-\kappa)s_i+\frac{\rho}{2}\sum_{i=1}^n\sum_{j=1}^n s_i s_j+\frac{1}{2}\sum_{i=1}^n\sum_{j\neq i}^n a_{ij}(\theta s_i s_j-\zeta)\right)} \\
&= \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\sum_{i=1}^n\left((1-\theta-\rho)\gamma_i-\kappa+\frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i} \sum_{G\in\mathcal{G}^n} e^{\frac{\eta}{2}\sum_{i=1}^n\sum_{j\neq i}^n a_{ij}(\theta s_i s_j-\zeta)} \\
&= \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\sum_{i=1}^n\left((1-\theta-\rho)\gamma_i-\kappa+\frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i} \prod_{i=1}^n\prod_{j\neq i}^n\left(1+e^{\frac{\eta}{2}(\theta s_i s_j-\zeta)}\right), \qquad (42)
\end{aligned}$$

where we have used the fact that

$$\sum_{G\in\mathcal{G}^n} e^{\eta\sum_{i<j}^n a_{ij}\sigma_{ij}} = \prod_{i=1}^n\prod_{j=i+1}^n\left(1+e^{\eta\sigma_{ij}}\right), \qquad (43)$$

for some symmetric $\sigma_{ij}=\sigma_{ji}$. Introducing the *Hamiltonian* [cf. e.g. Grimmett, 2010]

$$\mathscr{H}^\eta(\mathbf{s}) \equiv \sum_{i=1}^n\left(\left((1-\theta-\rho)\gamma_i-\kappa+\frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i+\sum_{j\neq i}^n\left(\frac{1}{\eta}\ln\left(1+e^{\frac{\eta}{2}(\theta s_i s_j-\zeta)}\right)\right)\right), \qquad (44)$$

we can write the partition function as follows

$$\mathscr{Z}^\eta = \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\mathscr{H}^\eta(\mathbf{s})}. \qquad (45)$$

With the Hamiltonian we can write the marginal distribution as follows

$$\begin{aligned}
\mu^\eta(\mathbf{s}) &= \frac{1}{\mathscr{Z}^\eta}\sum_{G\in\mathcal{G}^n} e^{\eta\Phi(\mathbf{s},G)} \\
&= \frac{1}{\mathscr{Z}_n^\eta} e^{\eta\sum_{i=1}^n\left((1-\theta-\rho)\gamma_i-\kappa+\frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i}\prod_{i=1}^n\prod_{j\neq i}^n\left(1+e^{\frac{\eta}{2}(\theta s_i s_j-\zeta)}\right) \\
&= \frac{1}{\mathscr{Z}^\eta} e^{\eta\mathscr{H}^\eta(\mathbf{s})}, \qquad (46)
\end{aligned}$$

where $\mathscr{H}^\eta(\mathbf{s})$ has been defined in Equation (52). The partition function is then given by

$$\mathscr{Z}^\eta = \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\mathscr{H}^\eta(\mathbf{s})}.$$

Next, we write the Hamiltonian as

$$\mathscr{H}^\eta(\mathbf{s}) = \sum_{i=1}^n\tilde{\psi}_i(s_i)+\sum_{i=1}^n\sum_{j\neq i}\tilde{\psi}_{ij}(s_i,s_j),$$

[20] See e.g. Grimmett [2010]; Wainwright and Jordan [2008].

where we have denoted by

$$\tilde{\psi}_i(s_i) = ((1-\theta-\rho)\gamma_i - \kappa)\, s_i,$$
$$\tilde{\psi}_{ij}(s_i, s_j) = \frac{\rho}{2} s_i s_j + \frac{1}{\eta}\ln\left(1 + e^{\frac{\eta}{2}(\theta s_i s_j - \zeta)}\right).$$

Hence, the joint distribution can be written as

$$\mu^\eta(\mathbf{s}) = \frac{1}{\mathscr{Z}^\eta} e^{\eta \mathscr{H}^\eta(\mathbf{s})} = \frac{1}{\mathscr{Z}^\eta} e^{\eta \sum_{i=1}^n \tilde{\psi}_i(s_i) + \sum_{i=1}^n \sum_{j\neq i}^n \tilde{\psi}_{ij}(s_i, s_j)} = \frac{1}{\mathscr{Z}^\eta} \prod_{i=1}^n \psi_i(s_i) \prod_{i=1}^n \prod_{j\neq i}^n \psi_{ij}(s_i, s_j),$$

where we have denoted by

$$\psi_i(s_i) = e^{\eta \tilde{\psi}_i(s_i)},$$
$$\psi_{ij}(s_i, s_j) = e^{\eta \tilde{\psi}_{ij}(s_i, s_j)}.$$

The belief propagation (BP) algorithm consists of an iterative updating procedure given by [Yedidia et al., 2001]

$$b_{i\to j}^{t+1}(s_j) = \frac{1}{c} \sum_{s_i \in \{-1,+1\}} \psi_{ij}(s_i, s_j)\psi_i(s_i) \prod_{k\neq i,j} b_{k\to i}^t(s_i), \tag{47}$$

with a normalizing factor c such that

$$\sum_{s_j \in \{-1,+1\}} b_{i\to j}^{t+1}(s_j) = 1.$$

Note that since either $s_j = +1$ or $s_j = -1$, Equation (47) can be written as

$$b_{i\to j}^{t+1}(+1) = \frac{1}{c} \sum_{s_i \in \{-1,+1\}} \psi_{ij}(s_i, +1)\psi_i(s_i) \prod_{k\neq i,j} b_{k\to i}^t(s_i),$$
$$b_{i\to j}^{t+1}(-1) = \frac{1}{c} \sum_{s_i \in \{-1,+1\}} \psi_{ij}(s_i, -1)\psi_i(s_i) \prod_{k\neq i,j} b_{k\to i}^t(s_i).$$

This can be written as

$$c \cdot b_{i\to j}^{t+1}(+1) = [\psi_{ij}(+1,+1)\psi_i(+1) \prod_{k\neq i,j} b_{k\to i}^t(+1) + \psi_{ij}(-1,+1)\psi_i(-1) \prod_{k\neq i,j} b_{k\to i}^t(-1)],$$
$$c \cdot b_{i\to j}^{t+1}(-1) = [\psi_{ij}(+1,-1)\psi_i(+1) \prod_{k\neq i,j} b_{k\to i}^t(+1) + \psi_{ij}(-1,-1)\psi_i(-1) \prod_{k\neq i,j} b_{k\to i}^t(-1)],$$

where

$$c \cdot b_{i\to j}^{t+1}(+1) + c \cdot b_{i\to j}^{t+1}(-1) = c.$$

since

$$b_{i\to j}^{t+1}(+1) + b_{i\to j}^{t+1}(-1) = 1.$$

When the $b_{i\to j}^{t+1}$ converge to a stationary state $b_{i\to j}^*$ then the marginal distributions can be approximated by

$$\mu_i(s_i) \simeq \psi_i(s_i) \prod_{k\neq i}^n b_{k\to i}^*(s_i),$$

$$\mu_{ij}(s_i,s_j) \simeq \psi_{ij}(s_i,s_j)\psi_i(s_i)\psi_j(s_j) \prod_{k\neq i,j}^n b_{k\to i}^*(s_i) \prod_{l\neq i,j}^n b_{l\to j}^*(s_j).$$

The marginals are non-negative, $\mu_i(s_i) \geq 0$ and $\mu_{ij}(s_i,s_j) \geq 0$, and satisfy $\sum_{s_i\in\{-1,+1\}} \mu_i(s_i) = 1$ and $\sum_{s_j\in\{-1,+1\}} \mu_{ij}(s_i,s_j) = \mu_i(s_i)$. Further, the (*Bethe*) free energy $\mathscr{F}^\eta = -\frac{1}{\eta}\ln \mathscr{Z}^\eta$ can be written as [Yedidia et al., 2001]

$$\mathscr{F}^\eta = \sum_{i=1}^n \sum_{j\neq i}^n \sum_{s_i,s_j\in\{-1,+1\}^2} \ln\left(\frac{\mu_{ij}(s_i,s_j)}{\psi_{ij}(s_i,s_j)\psi_i(s_i)\psi_j(s_j)}\right) - (n-1)\sum_{i=1}^n \sum_{s_i\in\{-1,+1\}} \mu_i(s_i)\ln\left(\frac{\mu_i(s_i)}{\psi_i(s_i)}\right).$$

From the free energy we can then compute the partition function

$$\mathscr{Z}^\eta = e^{-\eta\mathscr{F}^\eta},$$

and the joint distribution

$$\mu^\eta(\mathbf{s}) = \frac{1}{\mathscr{Z}^\eta} e^{\eta\mathscr{H}^\eta(\mathbf{s})} = e^{\eta(\mathscr{H}^\eta(\mathbf{s})+\mathscr{F}^\eta)}.$$

C. Proofs

Proof of Proposition 1. The potential has the property that $\Phi(s_i',\mathbf{s}_{-i},G)-\Phi(\mathbf{s},G) = \pi_i(s_i',\mathbf{s}_{-i},G),G)-\pi_i(\mathbf{s},G) = (1-\theta-\rho)\gamma_i(s_i'-s_i)+\theta(s_i'-s_i)\sum_{j=1}^n a_{ij}s_j + \rho(s_i'-s_i)\sum_{j\neq i}^n s_j - \kappa(s_i'-s_i)$ and that $\Phi(\mathbf{s},G\pm ij)-\Phi(\mathbf{s},G) = \pi_i(\mathbf{s},G\pm ij)-\pi_i(\mathbf{s},G) = \pm(\theta s_i s_j - \zeta)$. □

Proof of Proposition 2. First, note that the embedded discrete time Markov chain is irreducible and aperiodic, and thus is ergodic and has a unique stationary distribution. Hence, also the continuous time Markov chain is ergodic and has a unique stationary distribution. The stationary distribution solves $\boldsymbol{\mu}^\eta\mathbf{Q} = \mathbf{0}$ with the transition rates matrix $\mathbf{Q} = (q(\boldsymbol{\omega},\boldsymbol{\omega}'))_{\boldsymbol{\omega},\boldsymbol{\omega}'\in\Omega}$ of Equation (4). This equation is satisfied when the probability distribution $\mu^\eta(\boldsymbol{\omega})$ satisfies the detailed balance condition [cf. e.g. Norris, 1998]

$$\mu^\eta(\boldsymbol{\omega})q(\boldsymbol{\omega},\boldsymbol{\omega}') = \mu^\eta(\boldsymbol{\omega}')q(\boldsymbol{\omega}',\boldsymbol{\omega}), \tag{48}$$

for all $\boldsymbol{\omega},\boldsymbol{\omega}'\in\Omega$. Observe that the detailed balance condition is trivially satisfied if $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}$ differ in more than one link or more than one action level. Hence, we consider only the case of link creation $G' = G+ij$ (and removal $G' = G-ij$) or an adjustment in action $s_i' \neq s_i$ for some $i \in \mathcal{N}$. For the case of link creation with a transition from $\boldsymbol{\omega} = (\mathbf{s},G)$ to $\boldsymbol{\omega}' = (\mathbf{s},G+ij)$ we can write the detailed balance condition as follows

$$\frac{1}{\mathscr{Z}^\eta} e^{\eta(\Phi(\mathbf{s},G)-m\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\eta\Phi(\mathbf{s},G+ij)}}{e^{\eta\Phi(\mathbf{s},G+ij)}+e^{\eta\Phi(\mathbf{s},G)}}\lambda = \frac{1}{\mathscr{Z}^\eta} e^{\eta(\Phi(\mathbf{s},G+ij)-(m+1)\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\eta\Phi(\mathbf{s},G)}}{e^{\eta\Phi(\mathbf{s},G)}+e^{\eta\Phi(\mathbf{s},G+ij)}}\xi.$$

This equality is trivially satisfied. A similar argument holds for the removal of a link with a transition from $\boldsymbol{\omega} = (\mathbf{s}, G)$ to $\boldsymbol{\omega}' = (\mathbf{s}, G - ij)$ where the detailed balance condition reads

$$\frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(\mathbf{s},G) - m\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\eta\Phi(\mathbf{s},G-ij)}}{e^{\eta\Phi(\mathbf{s},G-ij)} + e^{\eta\Phi(\mathbf{s},G)}} \xi = \frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(\mathbf{s},G-ij) - (m-1)\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\eta\Phi(\mathbf{s},G)}}{e^{\eta\Phi(\mathbf{s},G)} + e^{\eta\Phi(\mathbf{s},G-ij)}} \lambda.$$

For a change in the agents' actions with a transition from $\boldsymbol{\omega} = (s_i, \mathbf{s}_{-i}, G)$ to $\boldsymbol{\omega}' = (s_i', \mathbf{s}_{-i}, G)$ we get the following detailed balance condition

$$\begin{aligned} &\frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(s_i,\mathbf{s}_{-i},G) - m\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\eta\Phi(s_i',\mathbf{s}_{-i},G)}}{e^{\eta\Phi(s_i,\mathbf{s}_{-i},G)} + e^{\eta\Phi(s_i',\mathbf{s}_{-i},G)}} \chi \\ &\qquad = \frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(s_i',\mathbf{s}_{-i},G) - m\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\eta\Phi(s_i,\mathbf{s}_{-i},G)}}{e^{\eta\Phi(s_i,\mathbf{s}_{-i},G)} + e^{\eta\Phi(s_i',\mathbf{s}_{-i},G)}} \chi. \end{aligned}$$

Hence, the probability measure $\mu^\eta(\boldsymbol{\omega})$ satisfies a detailed balance condition of Equation (48) and therefore is the stationary distribution of the Markov chain with transition rates $q(\boldsymbol{\omega}, \boldsymbol{\omega}')$. □

Lemma 1. *Conditional on the action profile* $\mathbf{s}$*, the probability of observing the network* G *is given by*

$$\mu^\eta(G|\mathbf{s}) = \prod_{i=1}^{n} \prod_{j=i+1}^{n} p_{ij}(s_i, s_j)^{a_{ij}} \left(1 - p_{ij}(s_i, s_j)\right)^{1-a_{ij}}, \tag{49}$$

where $p_{ij}(s_i, s_j) = \frac{e^{\eta(\theta s_i s_j - \zeta)}}{1+e^{\eta(\theta s_i s_j - \zeta)}}$.

Proof of Lemma 1. We first compute the *partition function* [cf. e.g. Grimmett, 2010; Wainwright and Jordan, 2008], which appears as the denominator in Equation (22), explicitly. We have that[21]

$$\begin{aligned} \mathscr{Z}^\eta &\equiv \sum_{G \in \mathcal{G}^n} \sum_{\mathbf{s} \in \{-1,+1\}^n} e^{\eta\Phi(\mathbf{s},G)} \\ &= \sum_{\mathbf{s} \in \{-1,+1\}^n} \sum_{G \in \mathcal{G}^n} e^{\eta\left(\sum_{i=1}^n((1-\theta-\rho)\gamma_i - \kappa)s_i + \frac{\rho}{2}\sum_{i=1}^n\sum_{j=1}^n s_i s_j + \sum_{i=1}^n\sum_{j=i+1}^n a_{ij}(\theta s_i s_j - \zeta)\right)} \\ &= \sum_{\mathbf{s} \in \{-1,+1\}^n} e^{\eta\sum_{i=1}^n\left((1-\theta-\rho)\gamma_i - \kappa + \frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i} \sum_{G \in \mathcal{G}^n} e^{\eta\sum_{i=1}^n\sum_{j=i+1}^n a_{ij}(\theta s_i s_j - \zeta)} \\ &= \sum_{\mathbf{s} \in \{-1,+1\}^n} e^{\eta\sum_{i=1}^n\left((1-\theta-\rho)\gamma_i - \kappa + \frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i} \prod_{i=1}^n \prod_{j=i+1}^n \left(1 + e^{\eta(\theta s_i s_j - \zeta)}\right), \end{aligned} \tag{50}$$

where we have used the fact that

$$\sum_{G \in \mathcal{G}^n} e^{\eta\sum_{i<j}^n a_{ij}\sigma_{ij}} = \prod_{i=1}^n \prod_{j=i+1}^n \left(1 + e^{\eta\sigma_{ij}}\right), \tag{51}$$

[21] Note that when the network is exogenous (i.e. when $\xi = \lambda = 0$) then in the limit of $\eta \to \infty$ the sum over all configurations $\mathbf{s} \in \{-1,+1\}^n$ is equivalent to summing over all max cuts of the underlying graph, whose enumeration is an NP hard problem (cf. A. Montanari, "Inference in Graphical Models", Stanford University, lecture notes, 2012).

for some $\sigma_{ij} = \sigma_{ji}$. Introducing the *Hamiltonian* [cf. e.g. Grimmett, 2010]

$$\mathscr{H}^\eta(\mathbf{s}) \equiv \sum_{i=1}^n \left(\left((1-\theta-\rho)\gamma_i - \kappa + \frac{\rho}{2}\sum_{j=1}^n s_j \right) s_i + \sum_{j=i+1}^n \left(\frac{1}{\eta} \ln\left(1 + e^{\eta(\theta s_i s_j - \zeta)}\right)\right)\right), \quad (52)$$

we can write the partition function as follows

$$\mathscr{Z}^\eta = \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta \mathscr{H}^\eta(\mathbf{s})}. \quad (53)$$

With the Hamiltonian we can write the marginal distribution as follows

$$\begin{aligned}
\mu^\eta(\mathbf{s}) &= \frac{1}{\mathscr{Z}^\eta} \sum_{G\in\mathcal{G}^n} e^{\eta\Phi(\mathbf{s},G)} \\
&= \frac{1}{\mathscr{Z}_n^\eta} e^{\eta\sum_{i=1}^n\left((1-\theta-\rho)\gamma_i-\kappa+\frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i} \prod_{i=1}^n \prod_{j=i+1}^n \left(1 + e^{\eta(\theta s_i s_j - \zeta)}\right) \\
&= \frac{1}{\mathscr{Z}^\eta} e^{\eta\mathscr{H}^\eta(\mathbf{s})}, \quad (54)
\end{aligned}$$

where $\mathscr{H}^\eta(\mathbf{s})$ has been defined in Equation (52). With the marginal distribution from Equation (54) we can write the conditional distribution as

$$\begin{aligned}
\mu^\eta(G|\mathbf{s}) = \frac{\mu^\eta(\mathbf{s},G)}{\mu^\eta(\mathbf{s})} &= \frac{e^{\eta\left(\sum_{i=1}^n\left((1-\theta-\rho)\gamma_i-\kappa+\frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i + \frac{\theta}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}s_i s_j - m\zeta\right)}}{e^{\eta\sum_{i=1}^n\left((1-\theta-\rho)\gamma_i-\kappa+\frac{\rho}{2}\sum_{j=1}^n s_j\right)s_i}\prod_{i=1}^n\prod_{j=i+1}^n\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right)} \\
&= \frac{e^{\eta\sum_{i<j}^n a_{ij}(\theta s_i s_j - \zeta)}}{\prod_{i=1}^n\prod_{j=i+1}^n\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right)} \\
&= \prod_{i<j} \frac{e^{\eta a_{ij}(\theta s_i s_j-\zeta)}}{1+e^{\eta(\theta s_i s_j-\zeta)}} \\
&= \prod_{i<j} \left(\frac{e^{\eta(\theta s_i s_j-\zeta)}}{1+e^{\eta(\theta s_i s_j-\zeta)}}\right)^{a_{ij}} \left(1 - \frac{e^{\eta(\theta s_i s_j-\zeta)}}{1+e^{\eta(\theta s_i s_j-\zeta)}}\right)^{1-a_{ij}} \\
&= \prod_{i<j} p_{ij}(s_i,s_j)^{a_{ij}} \left(1 - p_{ij}(s_i,s_j)\right)^{1-a_{ij}}. \quad (55)
\end{aligned}$$

Hence, conditional on the action choices $\mathbf{s}$, we obtain the likelihood of an inhomogeneous random graph with link probability

$$p_{ij}(s_i,s_j) = \frac{e^{\eta(\theta s_i s_j-\zeta)}}{1+e^{\eta(\theta s_i s_j-\zeta)}}. \quad (56)$$

□

In the following we provide an explicit computation of the partition function introduced in Equation (50).

Lemma 2. *The partition function of Equation (50) is given by*

$$\mathscr{Z}^\eta = \sum_{k=0}^{n} \sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} e^{\eta(1-\theta-\rho)(2k-n)}$$
$$\times\, e^{\eta(\frac{\rho}{2}(n+2l(k,j)-\binom{n}{2}))-\kappa(n-2(n_++k-2j))} \left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}} \left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}, \quad (57)$$

where

$$l(k,j) = \frac{n^2 + (2(2j-k)-1)n + 2(2j-k)^2 - 2(n+2(2j-k)-n_+)n_+}{2},$$

and $n_+ = \#(\{\gamma_i = 1 : i = 1,\ldots,n\})$.

Note that, while the evaluation of the partition function in Equation (50) requires the computation of a sum with 2^n terms, the partition function in Equation (57) requires the evaluation of only $\frac{1}{2}(n_++1)(2(n+1)-n_+) = O(n)$ terms. With Equation (57) the marginal distribution $\mu^\eta(\mathbf{s})$ in Equation (54) can then be efficiently computed.

Proof of Lemma 2. Assume w.l.o.g. that the agents are ordered such that $\gamma_1 = \ldots \gamma_{n_+} = +1$ and $\gamma_{n_++1} = \ldots \gamma_n = -1$, with $0 \le n_+ \le n$. Let us consider all configurations $\mathbf{s} \in \{-1,+1\}^n$ for which there $k = 0,\ldots,n$ agents with $s_i = \gamma_i$. For a given k, there are $\binom{n_+}{j}$ ways to select j agents from n_+ choosing $s_i = \gamma_i = +1$, and there are $\binom{n-n_+}{k-j}$ ways to select $k-j$ agents from n_- choosing $s_i = \gamma_i = -1$, for each $j = 0,\ldots,\min\{k,n_+\}$. Hence, there are

$$\sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j}$$

ways to obtain alignments of $\boldsymbol{\gamma}$ and $\mathbf{s}$ such that $\sum_{i=1}^n s_i\gamma_i = k-(n-k) = 2k-n$.

Next, we consider the products s_is_j. Since all the j agents in n_+ with $s_i = +1$ choose the same action $+1$, and all the $k-j$ agents in n_- with $s_i = -1$ choose the same action -1 we obtain

$$l(k,j) = \binom{j}{2} + \binom{k-j}{2} + \binom{n_+-j}{2} + \binom{n-n_+-(k-j)}{2} + (n_+-j)(k-j) + j(n-n_+-(k-j))$$

pairs whose product of actions gives $s_is_j = +1$. The first term in the equation above counts all pairs of agents with action $+1$ in the first set (with all $\gamma_i = +1$), the second all pairs of agents with action -1 in the second set (with all $\gamma_i = -1$), the third term the pairs of agents with action -1 in the first set (with all $\gamma_i = +1$), the fourth term the pairs of agents with action $+1$ in the second set (with all $\gamma_i = -1$), the fifth term counts the pairs with agents in the first set who choose action -1 and the agents in the second set who chose action -1, while the last term counts the pairs with agents in the first set who choose action $+1$ and agents in the second set who also choose action $+1$.

We can further simplify $l(k,j)$ to

$$l(k,j) = \frac{n^2 + (2(2j-k)-1)n + 2(2j-k)^2 - 2(n+2(2j-k)-n_+)n_+}{2}.$$

Then we can write

$$
\begin{aligned}
\mathscr{Z}^\eta &= \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\mathscr{H}^\eta(\mathbf{s})} \\
&= \sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} \exp\Bigg\{ \eta(1-\theta-\rho)(2k-n) \\
&\quad -\kappa\left(n-2(n_+ + k - 2j)\right) + \frac{\rho}{2}\left(n + 2l(k,j) - \binom{n}{2}\right) \\
&\quad + \frac{l(k,j)}{\eta}\ln\left(1+e^{\eta(\theta-\zeta)}\right) + \frac{\binom{n}{2}-l(k,j)}{\eta}\ln\left(1+e^{-\eta(\theta+\zeta)}\right)\Bigg\} \\
&= \sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} e^{\eta(1-\theta-\rho)(2k-n)} \\
&\quad \times e^{\eta(\frac{\rho}{2}\left(n+2l(k,j)-\binom{n}{2}\right))-\kappa(n-2(n_++k-2j))}\left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}},
\end{aligned}
$$

where $n_+ = \#(\{\gamma_i = 1 : i = 1,\ldots,n\})$. □

Proof of Proposition 3. Knowing the partition function allows us to compute the expected number of links, m, as

$$
\mathbb{E}^\eta(m) = \sum_{G\in\mathcal{G}^n}\sum_{\mathbf{s}\in\{-1,+1\}^n} m\mu^\eta(\mathbf{s},G) = \frac{1}{\mathscr{Z}^\eta}\sum_{G\in\mathcal{G}^n}\sum_{\mathbf{s}\in\{-1,+1\}^n} \underbrace{me^{\eta\Phi(\mathbf{s},G)}}_{-\frac{1}{\eta}\frac{\partial}{\partial\zeta}e^{\eta\Phi(\mathbf{s},G)}} = -\frac{1}{\eta}\frac{1}{\mathscr{Z}^\eta}\frac{\partial\mathscr{Z}^\eta}{\partial\zeta}. \tag{58}
$$

With Equations (50) and (58) we then can compute the expected number of links as

$$
\begin{aligned}
\mathbb{E}^\eta(m) &= \frac{1}{\eta}\frac{1}{\mathscr{Z}^\eta}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} e^{\eta(1-\theta-\rho)(2k-n)} e^{\eta(\frac{\rho}{2}\left(n+2l(k,j)-\binom{n}{2}\right))-\kappa(n-2(n_++k-2j))} \\
&\quad \times \left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}\left(\frac{l(k,j)}{1+e^{-\eta(\theta-\zeta)}} + \frac{\binom{n}{2}-l(k,j)}{1+e^{\eta(\theta+\zeta)}}\right).
\end{aligned}
$$

For $\theta = \rho = 0$ this simplifies to

$$\mathbb{E}^{\eta}(m) = \frac{1}{\eta}\frac{1}{\mathscr{Z}^{\eta}}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}e^{\eta(1-\rho)(2k-n)}$$
$$\times\left(1+e^{-\eta\zeta}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta\zeta}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}\left(\frac{l(k,j)}{1+e^{\eta\zeta}}+\frac{\binom{n}{2}-l(k,j)}{1+e^{\eta\zeta}}\right)$$
$$= \frac{1}{\eta}\frac{1}{\mathscr{Z}^{\eta}}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}\binom{n}{2}e^{\eta(2k-n)}\left(1+e^{-\eta\zeta}\right)^{\frac{\binom{n}{2}}{\eta}}\frac{1}{1+e^{\eta\zeta}}$$
$$= \frac{1}{\eta}\frac{1}{\mathscr{Z}^{\eta}}\binom{n}{2}\left(1+e^{-\eta\zeta}\right)^{\frac{\binom{n}{2}}{\eta}}\frac{1}{1+e^{\eta\zeta}}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}e^{\eta(2k-n)}$$
$$= \frac{1}{\mathscr{Z}^{\eta}}\frac{e^{-\eta n}}{\eta\pi(1+e^{\eta\zeta})}\binom{n}{2}\left(1+e^{-\eta\zeta}\right)^{\frac{\binom{n}{2}}{\eta}}$$
$$\times\left(\pi\left(1+e^{2\eta}\right)^{n}-e^{2(n+1)\eta}\sin(n\pi)\Gamma(n+1)\,{}_2F_1\left(1,1;n+2;-e^{2\eta}\right)\right),$$

and one can show that for $\zeta > 0$ we have that $\lim_{\eta\to\infty}\mathbb{E}^{\eta}(m) = 0$. □

Proof of Proposition 4. For the average action level $\bar{s} = \frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\mathbf{u}^{\top}\mathbf{s}$ we have that

$$\mathbb{E}^{\eta}(\bar{s}) = \sum_{\mathbf{s}\in\{-1,+1\}^n}\bar{s}\mu^{\eta}(\mathbf{s})$$
$$= \frac{1}{\mathscr{Z}^{\eta}}\sum_{\mathbf{s}\in\{-1,+1\}^n}\frac{1}{n}\mathbf{u}^{\top}\mathbf{s}e^{\eta\mathscr{H}^{\eta}(\mathbf{s})}$$
$$= \frac{1}{\mathscr{Z}^{\eta}}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}\frac{j+(n-n_+-(k-j))-(n_+-j+(k-j))}{n}$$
$$\times e^{\eta(1-\theta-\rho)(2k-n)}e^{\eta(\frac{\rho}{2}(n+2l(k,j)-\binom{n}{2}))-\kappa(n-2(n_++k-2j))}\left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}$$
$$= \frac{1}{\mathscr{Z}^{\eta}}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}\frac{n+4j-2(n_++k)}{n}$$
$$\times e^{\eta(1-\theta-\rho)(2k-n)}e^{\eta(\frac{\rho}{2}(n+2l(k,j)-\binom{n}{2}))-\kappa(n-2(n_++k-2j))}\left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}.$$

□

Proof of Proposition 5. Recall that the potential function in Proposition 1 is given by

$$\Phi(\mathbf{s},G) = (1-\theta-\rho)\sum_{i=1}^{n}\gamma_i s_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\theta s_i s_j - \zeta) + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}s_i s_j - \kappa\sum_{i=1}^{n}s_i$$
$$= \sum_{i=1}^{n}\left((1-\theta-\rho)\gamma_i + \frac{\rho}{2}\sum_{j\neq i}^{n}s_j - \kappa\right)s_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\theta s_i s_j - \zeta). \tag{59}$$

Note that only the last term in Equation (59) depends on the network (through the entries of the adjacency matrix elements a_{ij}). In particular, the term $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}s_i s_j$ is maximized over $s_i, s_j \in \{-1, +1\}$ for $a_{ij} = 1$ iff $s_i = s_j$. Similarly, the term $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\theta s_i s_j - \zeta)$ is maximized over $s_i, s_j \in \{-1, +1\}$ for $a_{ij} = 1$ iff $s_i = s_j$ and $\zeta < \theta$, while $a_{ij} = 0$ otherwise. The latter implies that the network must be either complete, K_n, empty, $\overline{K}_n$, or composed of two disconnected cliques, $K_{n_1} \cup K_{n-n_1}$, in which all agents in the same clique chose the same action.

Consider first the case of $\theta < \zeta$. Then the stochastically stable network is empty, $\overline{K}_n$ and the potential function simplifies to

$$\Phi(\mathbf{s}, \overline{K}_n) = (1-\theta-\rho)\sum_{i=1}^{n} s_i\gamma_i + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} s_i s_j - \kappa\sum_{i=1}^{n} s_i.$$

Observe that the first term is maximized if $s_i = \gamma_i$, the second term is maximized if $s_i = s_j$ for all i and j, while the last term is maximized if $s_i = -1$ for all i. The second and third terms are jointly maximized if all agents choose $s_i = -1$. We thus need to consider only two possible cases for the action profiles. All agents i choose $s_i = -1$ or all agent choose $s_i = \gamma_i$. We can ignore configurations different from the above in which some agent i with $\gamma_i = +1$ would choose an action $s_i = -1$. This is because if the potential would be higher in such a configuration, then it would be even higher in the case where all agents choose $s_i = -1$.

In the case of all agents choosing the action $s_i = -1$ the potential is given by

$$\Phi((-1,\ldots,-1), \overline{K}_n) = (1-\theta-\rho)((n-n_+)-n_+) + \frac{\rho n(n-1)}{2} + \kappa n.$$

In the case of all agents choosing the action $s_i = \gamma_i$ the potential is given by

$$\begin{aligned}\Phi(\boldsymbol{\gamma}, \overline{K}_n) &= (1-\theta-\rho)n + \frac{\rho}{2}\left(n_+((n_+-1)-(n-n_+)) + (n-n_+)((n-n_+-1)-n_+)\right) \\ &\quad - \kappa(n_+ - (n-n_+)) \\ &= (1-\theta-\rho)n + \frac{\rho}{2}\left((n-2n_+)^2 - n\right) - \kappa(2n_+ - n).\end{aligned}$$

Solving $\Phi((-1,\ldots,-1), \overline{K}_n) = \Phi(\boldsymbol{\gamma}, \overline{K}_n)$ for θ yields the threshold

$$\tilde{\theta} = 1 - \kappa - \rho(n - n_+ + 1).$$

For $\theta > \tilde{\theta}$ the stochastically stable state will be the one in which all agents choose the action $s_i = -1$, while for $\theta < \tilde{\theta}$ all agents choose the action $s_i = \gamma_i$. A similar threshold can be computed for ρ yielding

$$\rho^* = \frac{1-\theta-\kappa}{n-n_+ + 1}.$$

In the following we assume that $n_+ < \frac{n}{2}$ and $\theta > \zeta$. First, consider two cliques, K_{n_+} and K_{n-n_+} of sizes n_+ and $n - n_+$, respectively, where the agents in K_{n_+} choose $s_i = \gamma_i = +1$, and the agents in K_{n-n_+} choose $s_i = \gamma_i = -1$. The potential function is then given by

$$\begin{aligned}
&\Phi(\boldsymbol{\gamma}, K_{n_+} \cup K_{n-n_+}) \\
&= (1-\theta-\rho)n + \frac{1}{2}(n_+(n_+-1) + (n-n_+)(n-n_+-1))(\theta-\zeta) \\
&+ \frac{\rho}{2}\left(n_+((n_+-1)-(n-n_+)) + (n-n_+)((n-n_+-1)-n_+)\right) - \kappa(n_+-(n-n_+)) \\
&= (1-\theta-\rho)n + \frac{1}{2}(n(n-1) - 2n_+(n-n_+))(\theta-\zeta) \\
&+ \frac{\rho}{2}\left(n_+(2n_+-n-1) + (n-n_+)(n-1-2n_+)\right) - \kappa(2n_+-n) \\
&= (1-\theta-\rho)n + \frac{1}{2}(n(n-1) - 2n_+(n-n_+))(\theta-\zeta) + \frac{\rho}{2}\left((n-2n_+)^2 - n\right) - \kappa(2n_+-n).
\end{aligned}$$

We next consider the potential in a union of cliques $K_{n_+-k} \cup K_{n-n_++k}$, obtained from disconnecting k nodes j from the clique K_{n_+} and connecting them to all nodes in the clique K_{n-n_+}, while choosing the action $s_j = -1$ with $\gamma_j = +1$, with $k = 0, \ldots, n_+$. This is illustrated in the left panel in Figure 2 for $k = 1$. The corresponding potential function is given by

$$\begin{aligned}
&\Phi(\mathbf{s}', K_{n_+-k} \cup K_{n-n_++k}) \\
&= (1-\theta-\rho)((n_+-k) - k + n - n_+) - \kappa((n_+-k) - (n-n_++k)) \\
&+ \frac{1}{2}((n_+-k)(n_+-k-1) + (n-n_++k)(n-n_++k-1))(\theta-\zeta) \\
&+ \frac{\rho}{2}\left((n_+-k)((n_+-k-1) - (n-n_++k)) + (n-n_++k)((n-n_++k-1) - (n_+-k))\right) \\
&= (1-\theta-\rho)(n-2k) - \kappa(2(n_+-k)-n) + \frac{\rho}{2}((n-2(n_+-k))^2 - n) \\
&+ \frac{1}{2}(n^2 - n(2n_+-2k+1) + 2(n_+-k)^2)(\theta-\zeta).
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\Phi(\mathbf{s}', K_{n_+-k} \cup K_{n-n_++k}) - \Phi(\boldsymbol{\gamma}, K_{n_+} \cup K_{n-n_+}) \\
&\qquad\qquad\qquad = k\left(2(\kappa - (1-\theta-\rho)) + (2\rho+\theta-\zeta)(n-2n_++k)\right). \quad (60)
\end{aligned}$$

Note that this is a convex function of k (see Figure 2).[22] A convex function attains its maximum at the boundaries. In particular, if $n_+ < \frac{n}{2}$ then $n - 2n_+ + k > 0$ for every $k = 0, \ldots, n_+$, and when $\kappa > 1-\theta-\rho$ then the difference in the potentials in Equation (60) is a strictly increasing function of k. The potential is then highest in the complete graph, K_n, in which all agents choose $s_i = -1$. In this case the potential is given by

$$\Phi((-1,\ldots,-1), K_n) = (1-\theta-\rho)(n-2n_+) + \frac{n(n-1)}{2}(\theta-\zeta) + \frac{\rho}{2}n(n-1) + \kappa n,$$

and we have that

$$\Phi((-1,\ldots,-1), K_n) - \Phi(\boldsymbol{\gamma}, K_{n_+} \cup K_{n-n_+}) = n_+(2(\theta+\kappa+\rho-1) + (n_+-n)(\zeta-\theta-2\rho)).$$

[22] Denote by $\Delta\Phi(k) \equiv \Phi(\mathbf{s}', K_{n_+-k} \cup K_{n-n_++k}) - \Phi(\boldsymbol{\gamma}, K_{n_+} \cup K_{n-n_+})$. Then $\frac{d\Delta\Phi(k)}{dk} = 2(\theta+\kappa+\rho-1) - (\zeta - \theta - 2\rho)(2k - 2n_+ + n)$ and $\frac{d^2\Delta\Phi(k)}{dk^2} = 2(2\rho+\theta-\zeta) > 0$ for $\theta > \zeta$. Further, note that if $\Delta\Phi(k)$ is convex, then also $\Phi(\mathbf{s}', K_{n_+-k} \cup K_{n-n_++k})$ is convex.

Solving $\Phi((-1,\dots,-1),K_n) = \Phi(\boldsymbol{\gamma}, K_{n_+} \cup K_{n-n_+})$ for θ yields the threshold

$$\theta^* = \frac{(n-n_+)(\zeta - 2\rho) + 2(1-\kappa-\rho)}{2+n-n_+}.$$

Hence, for $\theta > \theta^*$ the stochastically stable state is the complete graph K_n in which all agents choose the action $s_i = -1$, while for $\theta < \theta^*$ it is the union of two cliques, $K_{n_+} \cup K_{n-n_+}$, in which all agents choose the action $s_i = \gamma_i$.

Next we consider the case of $n_+ > \frac{n}{2}$ and $\theta > \zeta$. Consider the complete graph K_n in which all agents choose $s_i = +1$. Then

$$\begin{aligned}\Phi((+1,\dots,+1),K_n) &= (1-\theta-\rho)(n_+ - (n-n_+)) + \frac{n(n-1)}{2}(\theta-\zeta) + \frac{\rho}{2}n(n-1) - \kappa n \\ &= (1-\theta-\rho)(2n_+ - n) + \frac{n(n-1)}{2}(\theta-\zeta) + \frac{\rho}{2}n(n-1) - \kappa n.\end{aligned}$$

Further, we have that

$$\Phi((+1,\dots,+1),K_n) - \Phi(\boldsymbol{\gamma}, K_{n_+} \cup K_{n-n_+}) = (n-n_+)(2(\theta-\kappa+(n_++1)\rho-1) + n_+(\theta-\zeta)).$$

Solving $\Phi((+1,\dots,+1),K_n) = \Phi(\boldsymbol{\gamma}, K_{n_+} \cup K_{n-n_+})$ for θ yields the threshold

$$\theta^{**} = \frac{n_+(\zeta-2\rho) + 2(1+\kappa-\rho)}{2+n_+}.$$

Next, note that

$$\Phi((-1,\dots,-1),K_n) - \Phi((+1,\dots,+1),K_n) = 2(1-\theta-\rho)(n-2n_+) + 2\kappa n, \tag{61}$$

For $n_+ < \frac{n}{2}$ Equation (61) is strictly positive. In contrast, for $n_+ > \frac{n}{2}$ we have that

$$\Phi((-1,\dots,-1),K_n) < \Phi((+1,\dots,+1),K_n)$$

if

$$\kappa < \frac{(1-\theta-\rho)(2n_+-n)}{n}.$$

With the above construction we have covered all possible partitions of agents into two cliques (including the complete and the empty graphs), and the actions they can choose. As these are the candidate potential maximizers, we have therefore identified the networks and action profiles that maximize the potential. This concludes the proof.

□

Proof of Proposition 6. In the following we compute an absorbing state of the Markov process introduced in Definition 2 in the limit of $\eta \to \infty$ characterizing the stochastically stable states. In such an absorbing state $(\mathbf{s}, G, \mathbf{p})$, given the beliefs $\mathbf{p}$ agents do not have an incentive to change their actions, $\mathbf{s}$, or links, G. Because differences in the potential correspond to differences in payoffs, this holds if the potential is maximized for such $(\mathbf{s}, G)$ given the beliefs $\mathbf{p}$. Conversely, the belief update equation (18) must be stationary given $(\mathbf{s}, G)$, that is, $p_i = f_i(\mathbf{s}, \mathbf{p}, G) = \varphi\frac{1}{d_i}\sum_{j=1}^n a_{ij}s_j + (1-\varphi)\frac{1}{d_i}\sum_{j=1}^n a_{ij}p_j$ for all $i = 1,\dots,n$. We then proceed by a guess and verify approach to check that the conditions for such a fixed point are satisfied.

We first consider the potential maximizing states $(\mathbf{s}, G)$ given the beliefs $\mathbf{p}$. The potential

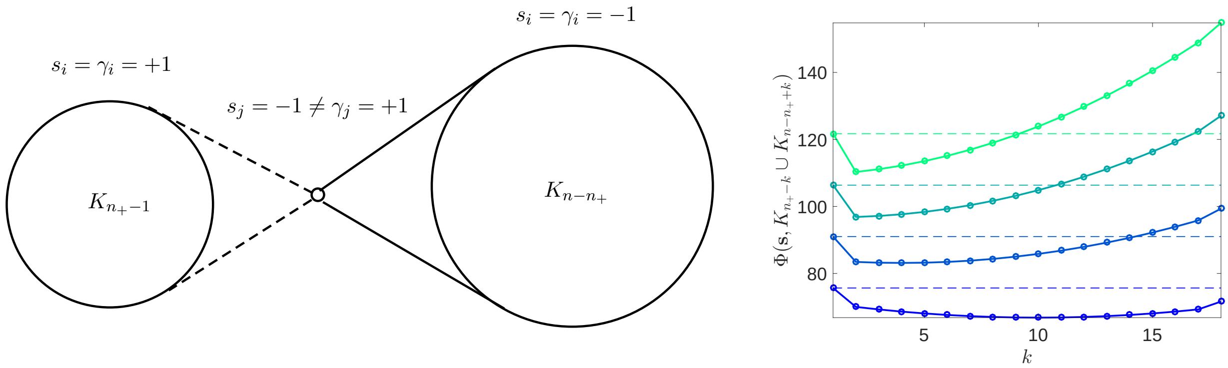

Figure 2: (Left panel) Illustration of two cliques, K_{n+} and K_{n-n_+} and the relocation of one node j from K_{n+} to K_{n-n_+}. (Right panel) The resulting potential for relocating node j from the clique K_{n_+} to the clique K_{n-n_+} for $\theta \in \{0.05, 0.075, 0.1, 0.125\}$, $n_+ = 17$, $n = 50$, $\rho = 0$ and $\zeta = 0.01$. The threshold is given by $\theta^* = 0.061$. For small values of $\theta < \theta^*$ the union of cliques $K_{n_+} \cup K_{n-n_+}$ ($j = 0$) has the highest potential, while for increasing values of θ the potential is highest for the complete graph K_n ($j = n_+ = 17$). We also see that the potential in a union of cliques $K_{n_+-k} \cup K_{n-n_++k}$ for $k = 1, \ldots, n_+ - 1$ is always smaller than the potential in the complete graph K_n or in the union of cliques $K_{n_+} \cup K_{n-n_+}$.

function can be written as

$$\Phi(\mathbf{s}, G, \mathbf{p}) = \tilde{\boldsymbol{\gamma}}^\top \mathbf{s} + \frac{\theta}{2} \mathbf{s}^\top \mathbf{A} \mathbf{s} - m\zeta,$$

where we have denoted by $\tilde{\gamma}_i = (1 - \theta - \rho)\gamma_i + \rho n p_i - \kappa$. For a given vector of beliefs, $\mathbf{p}$, the scalar product $\langle \tilde{\boldsymbol{\gamma}}, \mathbf{s} \rangle = \tilde{\boldsymbol{\gamma}}^\top \mathbf{s}$ is maximized for $s_i = \text{sign}(\tilde{\gamma}_i)$, and the quadratic form $\mathbf{s}^\top \mathbf{A} \mathbf{s}$ is maximized for $a_{ij} = 1$ iff $\text{sign}(s_i) = \text{sign}(s_j)$, or equivalently $\text{sign}(\tilde{\gamma}_i) = \text{sign}(\tilde{\gamma}_j)$ in the case of $\zeta < 1$. This implies that the stochastically stable network must be either complete, empty or composed of two cliques, where in each clique the agents choose the same actions.

The stationary beliefs for a given $(\mathbf{s}, G)$ are the solution to the following equation

$$\mathbf{p} = \mathbf{D}^{-1} \mathbf{A} \left(\varphi \mathbf{s} + (1 - \varphi) \mathbf{p} \right),$$

where $\mathbf{D} = \text{diag}(d_1, \ldots, d_n)$ is the diagonal matrix of the agents' degrees. This equation can be written as

$$\left(\mathbf{I}_n - (1 - \varphi) \mathbf{D}^{-1} \mathbf{A} \right) \mathbf{p} = \varphi \mathbf{D}^{-1} \mathbf{A} \mathbf{s},$$

with the solution

$$\mathbf{p} = \varphi \left(\mathbf{I}_n - (1 - \varphi) \mathbf{D}^{-1} \mathbf{A} \right)^{-1} \mathbf{D}^{-1} \mathbf{A} \mathbf{s}.$$

In the following we denote the row stochastic matrix $\mathbf{D}^{-1}\mathbf{A}$ by $\mathbf{Q}$.[23] Then we can write the stationary beliefs as follows

$$\mathbf{p} = \varphi \left(\mathbf{I}_n - (1 - \varphi) \mathbf{Q} \right)^{-1} \mathbf{Q} \mathbf{s}. \tag{62}$$

Hence, we know that the absorbing state $(\mathbf{s}, G, \mathbf{p})$ must satisfy $s_i = \text{sign}((1 - \theta - \rho)\gamma_i + \rho n p_i - \kappa)$

[23] Note that the stochastic matrix $\mathbf{Q}$ characterizes a random walk on a graph G with adjacency matrix $\mathbf{A}$ [cf. Van Mieghem, 2011]. The left eigenvector of $\mathbf{Q}$ belonging to the eigenvalue $\lambda = 1$ is the invariant measure of this random walk. The convergence of the random walk to the stationary state depends on the spectral gap of $\mathbf{Q}$.[24] Moreover, the stochastic matrix $\mathbf{Q}$ can be expressed in terms of the graph Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$ as $\mathbf{Q} = \mathbf{I}_n - \mathbf{D}^{-1}\mathbf{L}$. Hence, the eigenvectors of $\mathbf{Q}$ corresponding to the eigenvalue λ are the eigenvectors of the normalized Laplacian $\mathbf{D}^{-1}\mathbf{L}$ for the eigenvalue $\mu = 1 - \lambda$ and $0 < \mu < 2$. The spectral gap of $\mathbf{Q}$ equals the second smallest eigenvalue of the normalized Laplacian $\mathbf{D}^{-1}\mathbf{L}$. An extensive discussion of the Laplacian and its applications can be found in Chung and Lu [2007] and Mesbahi and Egerstedt [2010].

and $a_{ij} = 1$ iff $\text{sign}(s_i) = \text{sign}(s_j)$ in the case of $\zeta < 1$, where the network must be either complete, empty or composed of two cliques, where in each clique the agents choose the same actions.

From the equation $s_i = \text{sign}((1-\theta-\rho)\gamma_i + \rho n p_i - \kappa) = \text{sign}((1-\theta-\rho)\gamma_i + \rho n s_i - \kappa)$ we see that $\gamma_i = -1$ and $s_i = +1$ only if $-(1-\theta-\rho) + \rho n - \kappa > 0$, or equivalently, $\theta > 1-(n+1)\rho + \kappa$. Similarly, for $\gamma_i = +1$ and $s_i = -1$ we must have that $\theta > 1-(n+1)\rho - \kappa$.

Further, from Equation (62) we know that the stationary beliefs must satisfy $p_i = \varphi \frac{1}{d_i} \sum_{j=1}^n a_{ij} s_j + (1-\varphi)\frac{1}{d_i}\sum_{j=1}^n a_{ij} p_j$. In a network where all connected agents choose the same action and have the same beliefs, this simplifies to $p_i = \varphi s_i + (1-\varphi)p_i$, and this equation is trivially satisfied for $p_i = s_i$.

When $p_i = s_i$ for all $i = 1, \ldots, n$ then the potential is given by

$$\Phi(\mathbf{s}, G) = (1-\theta-\rho)\sum_{i=1}^n \gamma_i s_i + \rho n \sum_{i=1}^n s_i^2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij}(\theta s_i s_j - \zeta) - \kappa \sum_{i=1}^n s_i.$$

In the following we assume that $n_+ \leq \frac{n}{2}$ and $\theta > \zeta$. First, consider two cliques, K_{n_+} and K_{n-n_+} of sizes n_+ and $n - n_+$, respectively, where the agents in K_{n_+} choose $s_i = \gamma_i = +1$, and the agents in K_{n-n_+} choose $s_i = \gamma_i = -1$. The potential function is then given by

$$\begin{aligned}\Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+}) &= (1-\theta-\rho)n + \frac{1}{2}(n(n-1) - 2n_+(n-n_+))(\theta-\zeta) + \rho n^2 - \kappa(n_+ - (n-n_+)) \\ &= (1-\theta-\rho)n + \frac{1}{2}(n(n-1) - 2n_+(n-n_+))(\theta-\zeta) + \rho n^2 - \kappa(2n_+ - n).\end{aligned}$$

We next consider the potential in a union of cliques $K_{n_+-k} \cup K_{n-n_++k}$, obtained from disconnecting k nodes j from the clique K_{n_+} and connecting it to all nodes in the clique K_{n-n_+}, while choosing the action $s_j = -1$ with $\gamma_j = +1$, with $k = 0, \ldots, n_+$. This is illustrated in the left panel in Figure 2 for $k = 1$. The corresponding potential function is given by

$$\begin{aligned}\Phi(\mathbf{s}', K_{n_+-k} \cup K_{n-n_++k}) &= (1-\theta-\rho)(n-2k) - \kappa(2(n_+-k) - n) + \rho n^2 \\ &\quad + \frac{1}{2}(n(n-1) + 2k(k+n) - 2n_+(n-n_++2k))(\theta-\zeta).\end{aligned}$$

It then follows that

$$\Phi(\mathbf{s}', K_{n_+-k} \cup K_{n-n_++k}) - \Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+}) = 2k(\kappa - (1-\theta-\rho)) + (n - 2n_+ + k)k(\theta-\zeta).$$

Note that this is a convex function of k (see Figure 2).[25] In particular, if $n_+ \leq \frac{n}{2}$ then $n - 2n_+ + k > 0$ for every $k = 0, \ldots, n_+$, and when $\kappa > 1-\theta-\rho$ then this is a strictly increasing function of k. The potential is then highest in the complete graph, K_n, in which all agents choose $s_i' = -1$. In this case the potential is given by

$$\Phi(\mathbf{s}', K_n) = (1-\theta-\rho)(n-2n_+) + \frac{n(n-1)}{2}(\theta-\zeta) + \rho n^2 + \kappa n,$$

[25] Denote by $\Delta\Phi(k) \equiv \Phi(\mathbf{s}', K_{n_+-k} \cup K_{n-n_++k}) - \Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+})$. Then $\frac{\partial \Delta\Phi(k)}{\partial k} = (\theta-\zeta)(n - 2(n_+ - k)) - 2(1-\theta-\kappa-\rho)$ and $\frac{\partial^2 \Delta\Phi(k)}{\partial k^2} = 2(\theta-\zeta) > 0$ for $\theta > \zeta$. Further, note that if $\Delta\Phi(k)$ is convex, then also $\Phi(\mathbf{s}', K_{n_+-k} \cup K_{n-n_++k})$ is convex.

and we have that

$$\Phi(\mathbf{s}', K_n) - \Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+}) = 2n_+(\kappa - (1-\theta-\rho)) + n_+(n-n_+)(\theta - \zeta).$$

Solving $\Phi(\mathbf{s}', K_n) = \Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+})$ for θ yields the threshold

$$\theta^* = \frac{(n-n_+)\zeta + 2(1-\kappa-\rho)}{2+n-n_+}.$$

Next we assume that $n_+ \geq \frac{n}{2}$ and $\theta > \zeta$. Consider the complete graph K_n in which all agents choose $s_i'' = +1$. Then

$$\Phi(\mathbf{s}'', K_n) = (1-\theta-\rho)(2n_+ - n) + \frac{n(n-1)}{2}(\theta - \zeta) + \rho n^2 - \kappa n.$$

Further, we have that

$$\Phi(\mathbf{s}'', K_n) - \Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+}) = 2(n-n_+)(\kappa + (1-\theta-\rho)) - n_+(n-n_+)(\theta - \zeta).$$

Solving $\Phi(\mathbf{s}'', K_n) = \Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+})$ for θ yields the threshold

$$\theta^* = \frac{n_+\zeta + 2(1+\kappa-\rho)}{2+n_+}.$$

We also have that

$$\Phi(\mathbf{s}', K_n) - \Phi(\mathbf{s}'', K_n) = 2(1-\theta-\rho)(n-2n_+) + 2\kappa n,$$

which is positive if $n_+ \leq \frac{n}{2}$. In contrast, if $n_+ \geq \frac{n}{2}$ then $\Phi(\mathbf{s}', K_n) < \Phi(\mathbf{s}'', K_n)$ if

$$\kappa < \frac{(1-\theta-\rho)(n-2n_+)}{n}.$$

Moreover, when $\theta < \zeta$ then the network that maximizes the potential will be empty, $\overline{K}_n$, and all agents $i = 1, \ldots, n$ choose their idiosyncratic preference, $s_i = \gamma_i$. This concludes the proof of the proposition. □

The following proposition characterizes the expected number of links for an arbitrary level of noise.

Proposition 7. *For a given vector of beliefs,* $\mathbf{p}$*, the expected number of links is given by*

$$\mathbb{E}^\eta(m) = \frac{1}{\mathscr{Z}^\eta} \sum_{\mathbf{s} \in \{-1,+1\}^n} e^{\eta \langle \tilde{\gamma}, \mathbf{s} \rangle h^\eta(\mathbf{s})}, \tag{63}$$

where $\tilde{\gamma}_i = (1-\theta-\rho)\gamma_i + \rho n p_i - \kappa$*, the partition function is*

$$\mathscr{Z}^\eta = \sum_{\mathbf{s} \in \{-1,+1\}^n} e^{\eta \langle \tilde{\gamma}, \mathbf{s} \rangle f^\eta(\mathbf{s})},$$

and

$$h^\eta(\mathbf{s}) = \frac{\left(e^{-\eta(\zeta+\theta)}+1\right)^{\alpha(\mathbf{s})}\left(e^{\eta(\theta-\zeta)}+1\right)^{\beta(\mathbf{s})-1}\left((\alpha(\mathbf{s})+\beta(\mathbf{s}))e^{\eta(\theta-\zeta)}+\alpha(\mathbf{s})+\beta(\mathbf{s})e^{2\eta\theta}\right)}{1+e^{\eta(\zeta+\theta)}},$$

$$f^\eta(\mathbf{s}) = \left(e^{-\eta(\zeta+\theta)}+1\right)^{\alpha(\mathbf{s})}\left(e^{\eta(\theta-\zeta)}+1\right)^{\beta(\mathbf{s})},$$

with $\alpha(\mathbf{s}) = n_+(\mathbf{s})(n-n_+(\mathbf{s}))$, $\beta(\mathbf{s}) = \frac{1}{2}(n(n-1)-2n_+(\mathbf{s})(n-n_+(\mathbf{s})))$ *and* $n_+(\mathbf{s}) = \#(\{s_i = 1 : i = 1,\dots,n\})$.

Before proceeding with the proof of Proposition 7 we state the following lemma which will be useful later.

Lemma 3. *For any* $\mathbf{s} \in \{-1,+1\}^n$ *we have that*

$$\prod_{i=1}^{n}\prod_{j=i+1}^{n}\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right) = \left(1+e^{-\eta(\theta+\zeta)}\right)^{n_+(n-n_+)}\left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{n(n-1)-2n_+(n-n_+)}{2}},$$

where $n_+ = \#(\{\gamma_i = 1 : i = 1,\dots,n\})$.

Proof of Lemma 3. In the following we denote by $f(\mathbf{s}) \equiv \prod_{i=1}^{n}\prod_{j=i+1}^{n}\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right)$ and $g(s_i,s_j) \equiv 1+e^{\eta(\theta-\zeta)}$. Then we can write

$$\begin{aligned}
f(\mathbf{s}) &= \prod_{i=1}^{n_+-1}\left(\prod_{j=i+1}^{n_+} g(s_i,s_j)\prod_{j=n_++1}^{n} g(s_i,s_j)\right)\prod_{j=n_++1}^{n} g(s_{n_+},s_j)\prod_{i=n_++1}^{n}\prod_{j=i+1}^{n} g(s_i,s_j)\\
&= \prod_{i=1}^{n_+-1}\left(\prod_{j=i+1}^{n_+} g(+1,+1)\prod_{j=n_++1}^{n} g(+1,-1)\right)\prod_{j=n_++1}^{n} g(+1,-1)\prod_{i=n_++1}^{n}\prod_{j=i+1}^{n} g(-1,-1)\\
&= \prod_{i=1}^{n_+-1} g(+1,+1)^{n_+-i}g(+1,-1)^{n-n_+}g(+1,-1)^{n-n_+}\prod_{i=n_++1}^{n} g(-1,-1)^{n-i}\\
&= g(+1,-1)^{n-n_+}g(+1,-1)^{(n-n_+)(n_+-1)}\prod_{i=1}^{n_+-1} g(+1,+1)^{n_+-i}\prod_{i=n_++1}^{n} g(-1,-1)^{n-i}\\
&= g(+1,-1)^{n-n_+}g(+1,-1)^{(n-n_+)(n_+-1)}g(+1,+1)^{\frac{n_+(n_+-1)}{2}}g(+1,+1)^{\frac{(n-n_+)(n-n_+-1)}{2}}\\
&= g(+1,-1)^{n_+(n-n_+)}g(+1,+1)^{\frac{n(n-1)-2n_+(n-n_+)}{2}}\\
&= \left(1+e^{-\eta(\theta+\zeta)}\right)^{n_+(n-n_+)}\left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{n(n-1)-2n_+(n-n_+)}{2}}.
\end{aligned}$$

This conclues the proof. □

Proof of Proposition 7. The partition function is given by

$$\begin{aligned}
\mathscr{Z}^\eta &\equiv \sum_{G\in\mathcal{G}^n}\sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\Phi(\mathbf{s},G)}\\
&= \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\sum_{i=1}^n\tilde{\gamma}_i s_i}\prod_{i=1}^{n}\prod_{j=i+1}^{n}\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right),
\end{aligned}$$

where we have denoted by

$$\tilde{\gamma}_i = (1-\theta-\rho)\gamma_i + \rho n p_i - \kappa.$$

The expected number of links is given by

$$\begin{aligned}\mathbb{E}^{\eta}(m) &= -\frac{1}{\eta}\frac{1}{\mathscr{Z}^{\eta}}\frac{\partial \mathscr{Z}^{\eta}}{\partial \zeta} \\ &= -\frac{1}{\eta}\frac{1}{\mathscr{Z}^{\eta}}\sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\sum_{i=1}^n \tilde{\gamma}_i s_i}\frac{\partial}{\partial \zeta}\prod_{i=1}^n\prod_{j=i+1}^n\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right).\end{aligned}$$

Denoting by $f^{\eta}(\mathbf{s}) \equiv \prod_{i=1}^n\prod_{j=i+1}^n\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right)$ from Lemma 3 it follows that

$$f^{\eta}(\mathbf{s}) = \left(1+e^{-\eta(\theta+\zeta)}\right)^{\alpha(\mathbf{s})}\left(1+e^{\eta(\theta-\zeta)}\right)^{\beta(\mathbf{s})},$$

where $\alpha(\mathbf{s}) = n_+(\mathbf{s})(n-n_+(\mathbf{s}))$, $\beta(\mathbf{s}) = \frac{1}{2}(n(n-1)-2n_+(\mathbf{s})(n-n_+(\mathbf{s})))$ and $n_+(\mathbf{s}) = \#(\{s_i = 1 : i = 1,\ldots,n\})$. Morover one can show that

$$h^{\eta}(\mathbf{s}) \equiv \frac{\partial f^{\eta}(\mathbf{s})}{\partial \zeta} = \frac{\left(e^{-\eta(\zeta+\theta)}+1\right)^{\alpha(\mathbf{s})}\left(e^{\eta(\theta-\zeta)}+1\right)^{\beta(\mathbf{s})-1}\left((\alpha(\mathbf{s})+\beta(\mathbf{s}))e^{\eta(\theta-\zeta)}+\alpha(\mathbf{s})+\beta(\mathbf{s})e^{2\eta\theta}\right)}{1+e^{\eta(\zeta+\theta)}},$$

and we can write

$$\mathbb{E}^{\eta}(m) = \frac{1}{\mathscr{Z}^{\eta}}\sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\langle\tilde{\gamma},\mathbf{s}\rangle h^{\eta}(\mathbf{s})},$$

where

$$\mathscr{Z}^{\eta} = \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\langle\tilde{\gamma},\mathbf{s}\rangle f^{\eta}(\mathbf{s})}.$$

□

Figure 3 shows the average degree $\bar{d} = 2m/n$ across different values of the linking cost $\zeta \in \{0,1,\ldots,3\}$ and $\eta = 1,2,3$ for given beliefs. The average degree is decreasing with the linking cost ζ, and the decline is higher, the larger is η.

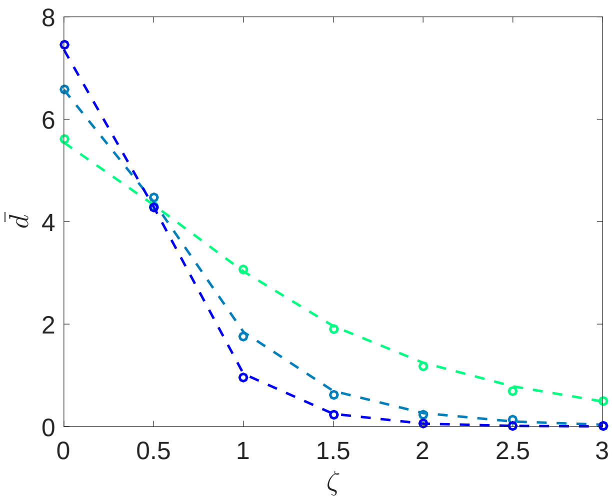

Figure 3: The average degree $\bar{d} = 2m/n$ across different values of the linking cost $\zeta \in \{0, 1, \ldots, 3\}$ and $\eta = 1, 2, 3$ conditional on beliefs. Dashed lines indicate the theoretical prediction of Equation (63) in Proposition 7 while circles indicate averages across 1000 numerical Monte Carlo simulations of the model.