Tinbergen Institute

Fundamental Mathematics

Linear Algebra Preparatory Test 1

Question 1
Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & -1 & -1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

a. Find the rank of A.

b. Find a basis of the row space $\text{RS}(A)$ and the column space $\text{CS}(A)$ of A.

c. Show that $\text{RS}(A)$ and $\text{CS}(A)$ are each planes in $\mathbb{R}^3$.

d. Find Cartesian equations for both $\text{RS}(A)$ and $\text{CS}(A)$.

e. Show that $\text{RS}(A)$ and $\text{CS}(A)$ are different subspaces of A.

f. Find a basis for the null space of A and verify the rank–nullity theorem.

g. Show that the basis vectors of the null space are orthogonal to the basis vectors of the row space of A.

h. Without solving the equations, determine whether the system of equations $Ax = b_1$ and $Ax = b_2$ are consistent. If the system is consistent, find the general solution.

i. If possible, express each of b_1 and b_2 as a linear combination of the columns of A.

Question 2
Let

$$A = \begin{pmatrix} 4 & 3 & -7 \\ 1 & 2 & 1 \\ 2 & 2 & -3 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

a. Show that v_1 is an eigenvector of A and find its corresponding eigenvalue.

b. Diagonalise the matrix A by finding an invertible matrice C and a diagonal matrix Λ such that $C^{-1}AC = \Lambda$. Check your answer without computing C^{-1}.

c. Compute $\det(A)$ from the eigenvalues.

d. Show that A is invertible.

e. Diagonalise A^{-1} without computing A^{-1}.

Question 3

Let

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}.$$

a. Starting with v_1, use the Gram–Schmidt algorithm to find an orthonormal basis $\{u_1, u_2, u_3\}$ of the subspace of $\mathbb{R}^4$ that is spanned by $\{v_1, v_2, v_3\}$.

b. Find a vector u_4 such that $\{u_1, u_2, u_3, u_4\}$ is an orthonormal basis of $\mathbb{R}^4$.

Question 4

Let A be an $m \times k$ matrix with real elements. Show that $A^T A$ cannot be negative definite.

Question 5

Orthogonally diagonalise the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and use this to sketch the curve $x^T A x = 3$ in the (x_1, x_2)-plane.